

LAGRANGIAN MECHANICS ON GENERALIZED LIE ALGEBROIDS

by
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Abstract

A solution for the Weinstein's Problem in the general framework of generalized Lie algebroids is the target of this paper. We present the *mechanical systems* called by use, *mechanical (ρ, η) -systems*, *Lagrange mechanical (ρ, η) -systems* or *Finsler mechanical (ρ, η) -systems* and we develop their geometries. We obtain the canonical (ρ, η) -semi(spray) associated to a mechanical (ρ, η) -system. The Lagrange mechanical (ρ, η) -systems are the spaces necessary to develop a Lagrangian formalism. We obtain the (ρ, η) -semispray associated to a regular Lagrangian L and external force F_e and we derive the equations of Euler-Lagrange type. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the Weinstein's problem.

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Contents

1	Introduction	2
2	Preliminaries	3
3	Natural and adapted basis	6
4	The lift of a differentiable curve	10
5	Remarkable Mod-endomorphisms	12
5.1	Projectors	13
5.2	The almost product structure	14
5.3	The almost tangent structure	15
6	Tensor d-fields. Distinguished linear (ρ, η)-connections	16
7	Mechanical systems	19
8	(ρ, η)-semisprays and (ρ, η)-sprays for mechanical (ρ, η)-systems	21
9	A Lagrangian formalism for Lagrange mechanical (ρ, η)-systems	28
	References	31

1 Introduction

The generalized Lie algebroid is a new notion necessary to obtain a new class of (linear) connections in Ehresmann sense.(see [1]) The notions of *IDS* and *EDS* for Lie algebroids presented in [3] was natural extended to generalized Lie algebroids in [2]. The identities of Cartan and Bianchi type presented in the final of the paper [2] emphasize the importance and the utility of the exterior differential calculus for generalized Lie algebroids. In particular there are obtained a new point of view over exterior differential calculus for Lie algebroids.

We know the **Weinstein's Problem**:

Develop a Lagrangian formalism directly on the given Lie algebroid similar to Klein's formalism for ordinary Lagrangian Mechanics (see [8]).

This problem was formulated by A. Weinstein in [17], where the author gave the theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the dual of a Lie algebroid and the Legendre transformation defined by a regular Lagrangian. In [10], P. Liberman showed that such a formalism is not possible if one consider the tangent bundle of a Lie algebroid as space for developing the theory. Using the prolongation of a Lie algebroid over a smooth map, E. Martinez solved the **Weinstein's Problem** in [11] (see also [6,9]).

In this paper we propose to solve the **Weinstein's Problem** in the general framework of generalized Lie algebroids.

In the Sections 3, 4, 5 and 6 we set up the basic notions and terminology. The Lagrange Geometry was studied by many authors. (see [4, 5, 7, 12, 13, 14, 15, 16]) In this paper we present the *mechanical systems* called by use, *mechanical* (ρ, η) -systems, *Lagrange mechanical* (ρ, η) -systems or *Finsler mechanical* (ρ, η) -systems. In Section 8 we study the geometry of mechanical (ρ, η) -systems. We present the *canonical* (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) . If $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to a connection Γ presented by I. Bucataru and R. Miron in [5]. Also, we present the *canonical* (ρ, η) -spray associated to mechanical system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) .

The Section 9 is dedicated to study the geometry of Lagrange mechanical (ρ, η) -systems. These mechanical systems are the spaces necessary to solve the **Weinstein's Problem** in the general framewok of generalized Lie algebroids. We determine and we study the (ρ, η) -semispray associated to a regular Lagrangian L and external force F_e which are applied on the total space of a generalized Lie algebroid and we derive the equations of Euler-Lagrange type. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the **Weinstein's Problem**, different by the Martinez's solution [11].

Finally, we obtain that the integral curves of the canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) are the (g, h) -lifts solutions for the equations of Euler-Lagrange type (9.10).

Using our theory, we obtain the following

Theorem *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g, h) -lifts are solutions for the equations of Euler-Lagrange type (9.10).*

As any Lie algebroid can be regarded as a particularly generalized Lie algebroid, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid $((TM, \tau_M, M), [\cdot, \cdot]_{TM}, (Id_{TM}, Id_M))$, to an arbitrary (generalized) Lie algebroid $((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta))$.

2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and **B^v** be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B}^v|$ so that M is paracompact and if $A \subseteq M$ is closed, then for any section u over A it exists $\tilde{u} \in \Gamma(E, \pi, M)$ so that $\tilde{u}|_A = u$. In the following, we consider only vector bundles with paracompact base.

Additionally, if $(E, \pi, M) \in |\mathbf{B}^v|$, $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$ -module. If $(\varphi, \varphi_0) \in \mathbf{B}^v((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$, then, using the operation

$$\begin{array}{ccc} \Gamma(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad} & \Gamma(E', \pi', M') \\ (f, u') & \mapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that $(\Gamma(E', \pi', M'), +, \cdot)$ is a $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \mapsto & \Gamma(\varphi, \varphi_0)u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0)u(y) = \varphi(u_{\varphi_0^{-1}(y)}),$$

for any $y \in M'$.

Let $M, N \in |\mathbf{Man}|$, $h \in Iso_{\mathbf{Man}}(M, N)$ and $\eta \in Iso_{\mathbf{Man}}(N, M)$.

We know (see [2, 3]) that if $(F, \nu, N) \in |\mathbf{B}^v|$ so that there exists

$$(\rho, \eta) \in \mathbf{B}^v((F, \nu, N), (TM, \tau_M, M))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \mapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

GLA₁. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$.

GLA₂. the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$ is a Lie $\mathcal{F}(N)$ -algebra,

GLA_3 . the **Mod**-morphism $\Gamma (Th \circ \rho, h \circ \eta)$ is a **LieAlg**-morphism of

$$\left(\Gamma (F, \nu, N), +, \cdot, [,]_{F,h} \right)$$

source and

$$(\Gamma (TN, \tau_N, N), +, \cdot, [,]_{TN})$$

target, then the triple $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$ is called generalized Lie algebroid.

In particular, if $h = Id_M = \eta$, then we obtain the definition of the Lie algebroid.

We can discuss about *the category GLA of generalized Lie algebroids*. (see [3])

Examples of objects of this category are presented in the paper [2].

Let $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$ be an object of the category **GLA**.

- Locally, for any $\alpha, \beta \in \overline{1, p}$, we set $[t_\alpha, t_\beta]_{F,h} = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha, \beta, \gamma \in \overline{1, p}$.

The real local functions $L_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma \in \overline{1, p}$ will be called the *structure functions of the generalized Lie algebroid* $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$.

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\ (\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}}) \end{array}$$

where $i, \tilde{i} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$\begin{aligned} (\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'} (\chi^{\tilde{i}}), z^{\alpha'} (\chi^{\tilde{i}}, z^\alpha)), \\ (x^i, y^i) &\longrightarrow (x^{\tilde{i}'} (x^i), y^{\tilde{i}'} (x^i, y^i)) \end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'} (\chi^{\tilde{i}}), z^{\tilde{i}'} (\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^\alpha t_\alpha \in \Gamma (F, \nu, N)$ is arbitrary, then

$$\begin{aligned} (2.1) \quad & \Gamma (Th \circ \rho, h \circ \eta) (z^\alpha t_\alpha) f (h \circ \eta (\varkappa)) = \\ & = \left(\theta_\alpha^{\tilde{i}} z^\alpha \frac{\partial f}{\partial \chi^{\tilde{i}}} \right) (h \circ \eta (\varkappa)) = \left((\rho_\alpha^i \circ h) (z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta (\varkappa)), \end{aligned}$$

for any $f \in \mathcal{F} (N)$ and $\varkappa \in N$.

The coefficients ρ_α^i respectively $\theta_\alpha^{\tilde{i}}$ change to $\rho_{\alpha'}^{\tilde{i}}$ respectively $\theta_{\alpha'}^{\tilde{i}}$ according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\tilde{i}} = \Lambda_\alpha^\alpha \rho_\alpha^i \frac{\partial x^{\tilde{i}}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}} = \Lambda_\alpha^\alpha \theta_\alpha^{\tilde{i}} \frac{\partial \mathcal{X}^{\tilde{i}}}{\partial \mathcal{X}^i},$$

where

$$\|\Lambda_{\alpha'}^\alpha\| = \|\Lambda_\alpha^\alpha\|^{-1}.$$

Remark 2.1 The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_\alpha^{\tilde{i}} \frac{\partial f}{\partial \mathcal{X}^{\tilde{i}}} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left(L_{\alpha\beta}^\gamma \circ h \right) \left(\rho_\gamma^k \circ h \right) = \left(\rho_\alpha^i \circ h \right) \frac{\partial \left(\rho_\beta^k \circ h \right)}{\partial x^i} - \left(\rho_\beta^j \circ h \right) \frac{\partial \left(\rho_\alpha^k \circ h \right)}{\partial x^j}.$$

We have the $\mathbf{B}^{\mathbf{v}}$ -morphism

$$(2.6) \quad \begin{array}{ccc} \pi^*(h^*F) & \hookrightarrow & F \\ \pi^*(h^*\nu) \downarrow & & \downarrow \nu \\ M & \xrightarrow{h \circ \pi} & N \end{array}$$

Let $\left(\pi^*(h^*F), Id_E \right)$ be the $\mathbf{B}^{\mathbf{v}}$ -morphism of $(\pi^*(h^*F), \pi^*(h^*\nu), E)$ source and (TE, τ_E, E) target, where

$$(2.7) \quad \begin{array}{ccc} \pi^*(h^*F) & \xrightarrow{\pi^*(h^*F)} & TE \\ Z^\alpha T_\alpha(u_x) & \longmapsto & (Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i}(u_x) \end{array}$$

Using the operation

$$\Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)^2 \xrightarrow{[\cdot]_{\pi^*(h^*F)}} \Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)$$

defined by

$$(2.8) \quad \begin{aligned} [T_\alpha, T_\beta]_{\pi^*(h^*F)} &= \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) T_\gamma, \\ [T_\alpha, fT_\beta]_{\pi^*(h^*F)} &= f \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) T_\gamma + (\rho_\alpha^i \circ h \circ \pi) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{\pi^*(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^*(h^*F)}, \end{aligned}$$

for any $f \in \mathcal{F}(E)$, it results that

$$\left((\pi^*(h^*F), \pi^*(h^*\nu), E), [\cdot]_{\pi^*(h^*F)}, \left(\pi^*(h^*F), Id_E \right) \right)$$

is a Lie algebroid.

3 Natural and adapted basis

We consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where (E, π, M) is a vector bundle and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid.

We take (x^i, y^a) as canonical local coordinates on (E, π, M) , where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Let

$$(x^i, y^a) \longrightarrow (x^{\check{i}}(x^i), y^{\check{a}}(x^i, y^a))$$

be a change of coordinates on (E, π, M) . Then the coordinates y^a change to $y^{\check{a}}$ by the rule:

$$(3.2) \quad y^{\check{a}} = M_a^{\check{a}} y^a.$$

For any sections

$$Z^\alpha T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*F), E)$$

and

$$Y^a \frac{\partial}{\partial y^a} \in \Gamma(VTE, \tau_E, E)$$

we obtain the section

$$\begin{aligned} Z^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a} &=: Z^\alpha \left(T_\alpha \oplus (\rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i} \right) + Y^a \left(0_{\pi^*(h^*F)} \oplus \frac{\partial}{\partial y^a} \right) \\ &= Z^\alpha T_\alpha \oplus \left(Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} \right) \in \Gamma\left(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E\right). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a} &= 0 \\ \Updownarrow \\ Z^\alpha T_\alpha = 0 \wedge Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} &= 0, \end{aligned}$$

it implies $Z^\alpha = 0$, $\alpha \in \overline{1, p}$ and $Y^a = 0$, $a \in \overline{1, r}$.

Therefore, the sections $\frac{\partial}{\partial \tilde{z}^1}, \dots, \frac{\partial}{\partial \tilde{z}^p}, \frac{\partial}{\partial \tilde{y}^1}, \dots, \frac{\partial}{\partial \tilde{y}^r}$ are linearly independent.

We consider the vector subbundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ of the vector bundle $(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E)$, for which the $\mathcal{F}(E)$ -module of sections is the $\mathcal{F}(E)$ -submodule of $(\Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E), +, \cdot)$, generated by the set of sections

$$(3.3) \quad \left(\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{y}^a} \right) \overset{put}{=} \left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a \right)$$

which is called the *natural* (ρ, η) -base.

The matrix of coordinate transformation on $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_{\alpha}^{\alpha'} \circ h \circ \pi & 0 \\ (\rho_a^i \circ h \circ \pi) \frac{\partial M_b^{\alpha'} \circ \pi}{\partial x_i} y^b & M_a^{\alpha'} \circ \pi \end{array} \right\|.$$

We have the following

Theorem 3.1 *Let $(\tilde{\rho}, Id_E)$ be the \mathbf{B}^v -morphism of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (TE, τ_E, E) target, where*

$$(3.5) \quad \begin{array}{c} (\rho, \eta) TE \xrightarrow{\tilde{\rho}} TE \\ \left(Z^{\alpha} \tilde{\partial}_{\alpha} + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \mapsto \left(Z^{\alpha} (\rho_{\alpha}^i \circ h \circ \pi) \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} \right) (u_x) \end{array}$$

Using the operation

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)^2 \xrightarrow{[\cdot]_{(\rho, \eta) TE}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

defined by

$$(3.6) \quad \begin{aligned} & \left[\left(Z_1^{\alpha} \tilde{\partial}_{\alpha} + Y_1^a \dot{\tilde{\partial}}_a \right), \left(Z_2^{\beta} \tilde{\partial}_{\beta} + Y_2^b \dot{\tilde{\partial}}_b \right) \right]_{(\rho, \eta) TE} \\ &= \left[Z_1^{\alpha} T_a, Z_2^{\beta} T_{\beta} \right]_{\pi^*(h^* F)} \oplus \left[\left(\rho_{\alpha}^i \circ h \circ \pi \right) Z_1^{\alpha} \frac{\partial}{\partial x^i} + Y_1^a \frac{\partial}{\partial y^a}, \right. \\ & \quad \left. \left(\rho_{\beta}^j \circ h \circ \pi \right) Z_2^{\beta} \frac{\partial}{\partial x^j} + Y_2^b \frac{\partial}{\partial y^b} \right]_{TE}, \end{aligned}$$

for any $\left(Z_1^{\alpha} \tilde{\partial}_{\alpha} + Y_1^a \dot{\tilde{\partial}}_a \right)$ and $\left(Z_2^{\beta} \tilde{\partial}_{\beta} + Y_2^b \dot{\tilde{\partial}}_b \right)$, we obtain that the couple

$$\left([\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right)$$

is a Lie algebroid structure for the vector bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

The Lie algebroid

$$\left(((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right),$$

is called the *Lie algebroid generalized tangent bundle*.

Remark 3.1 The following equalities hold good:

$$(3.7) \quad \begin{aligned} \left[\tilde{\partial}_{\alpha}, \tilde{\partial}_{\beta} \right]_{(\rho, \eta) TE} &= L_{\alpha\beta}^{\gamma} \circ h \circ \pi \cdot \tilde{\partial}_{\gamma} \\ \left[\tilde{\partial}_{\alpha}, \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= 0_{(\rho, \eta) TE} \\ \left[\tilde{\partial}_a, \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= 0_{(\rho, \eta) TE} \end{aligned}$$

We consider the $\mathbf{B}^{\mathbf{V}}$ -morphism $((\rho, \eta) \pi!, Id_E)$ given by the commutative diagram

$$(3.8) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^* (h^* F) \\ (\rho, \eta) \tau_E \downarrow & & \downarrow pr_1 \\ E & \xrightarrow{id_E} & E \end{array}$$

This is defined as:

$$(3.9) \quad (\rho, \eta) \pi! \left(\left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \right) = (Z^\alpha T_\alpha) (u_x),$$

for any $\left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Using the $\mathbf{B}^{\mathbf{V}}$ -morphisms (2.6) and (3.7) we obtain the *tangent* (ρ, η) -application $((\rho, \eta) T\pi, h \circ \pi)$ of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (F, ν, N) target.

Definition 3.1 The kernel of the tangent (ρ, η) -application is written

$$(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and it is called *the vertical subbundle*.

We remark that the set $\left\{ \dot{\tilde{\partial}}_a, a \in \overline{1, r} \right\}$ is a base of the $\mathcal{F}(E)$ -module

$$(\Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot).$$

Proposition 3.1 *The short sequence of vector bundles*

$$(3.10) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta) TE & \xrightarrow{i} & (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^* (h^* F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E \end{array}$$

is exact.

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle (E, π, M) , i. e. a **Man**-morphism $(\rho, \eta) \Gamma$ of $(\rho, \eta) TE$ source and $V(\rho, \eta) TE$ target defined by

$$(3.11) \quad (\rho, \eta) \Gamma \left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) = (Y^a + (\rho, \eta) \Gamma_\alpha^a Z^\alpha) \dot{\tilde{\partial}}_a (u_x),$$

so that the $\mathbf{B}^{\mathbf{V}}$ -morphism $((\rho, \eta) \Gamma, Id_E)$ is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.12) \quad (\rho, \eta) \Gamma_\gamma^a = M_a^a \circ \pi \left[\rho_\gamma^i \circ (h \circ \pi) \frac{\partial M_b^a \circ \pi}{\partial x^i} y^b + (\rho, \eta) \Gamma_\gamma^a \right] \Lambda_\gamma^\gamma \circ (h \circ \pi).$$

The kernel of the $\mathbf{B}^{\mathbf{V}}$ -morphism $((\rho, \eta) \Gamma, Id_E)$ is written $(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ and is called the *horizontal vector subbundle*.

We remark that the horizontal and the vertical vector subbundles are interior differential systems of the Lie algebroid generalized tangent bundle. (see [4])

We put the problem of finding a base for the $\mathcal{F}(E)$ -module

$$(\Gamma(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$$

of the type

$$\frac{\delta}{\delta \bar{z}^\alpha} = Z_\alpha^\beta \tilde{\partial}_\alpha + Y_\alpha^a \dot{\tilde{\partial}}_a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.13) \quad \begin{aligned} \Gamma((\rho, \eta)\pi!, Id_E) \left(\frac{\delta}{\delta \bar{z}^\alpha} \right) &= T_\alpha, \\ \Gamma((\rho, \eta)\Gamma, Id_E) \left(\frac{\delta}{\delta \bar{z}^\alpha} \right) &= 0. \end{aligned}$$

Then we obtain the sections

$$(3.14) \quad \frac{\delta}{\delta \bar{z}^\alpha} = \tilde{\partial}_\alpha - (\rho, \eta)\Gamma_\alpha^a \dot{\tilde{\partial}}_a = T_\alpha \oplus \left((\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta)\Gamma_\alpha^a \dot{\partial}_a \right).$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base

$$\left(\frac{\delta}{\delta \bar{z}^\alpha}, \frac{\partial}{\partial \bar{y}^a} \right) \stackrel{put}{=} \left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a \right)$$

will be called the *adapted* (ρ, η) -base.

Remark 3.2 The following equality holds good

$$(3.15) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\partial}_\alpha \right) = (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta)\Gamma_\alpha^a \dot{\partial}_a,$$

where $(\partial_i, \dot{\partial}_a)$ is the natural base for the $\mathcal{F}(E)$ -module $(\Gamma(TE, \tau_E, E), +, \cdot)$.

Moreover, if $(\rho, \eta)\Gamma$ is the (ρ, η) -connection associated to the connection Γ (see [1]), then we obtain

$$(3.16) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right) = (\rho_\alpha^i \circ h \circ \pi) \delta_i,$$

where $(\delta_i, \dot{\partial}_a)$ is the adapted base for the $\mathcal{F}(E)$ -module $(\Gamma(TE, \tau_E, E), +, \cdot)$.

Theorem 3.2 *The following equality holds good*

$$(3.17) \quad \left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta \right]_{(\rho, \eta)TE} = L_{\alpha\beta}^\gamma \circ (h \circ \pi) \tilde{\delta}_\gamma + (\rho, \eta, h) \mathbb{R}_{\alpha\beta}^a \dot{\tilde{\partial}}_a,$$

where

$$(3.18) \quad \begin{aligned} (\rho, \eta, h) \mathbb{R}_{\alpha\beta}^a &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\beta \right) ((\rho, \eta)\Gamma_\alpha^a) \\ &\quad - \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right) ((\rho, \eta)\Gamma_\beta^a) + (L_{\alpha\beta}^\gamma \circ h \circ \pi) (\rho, \eta)\Gamma_\gamma^a, \end{aligned}$$

Moreover, we have:

$$(3.19) \quad \left[\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} = \Gamma(\tilde{\rho}, Id_E) \left(\dot{\tilde{\partial}}_b \right) ((\rho, \eta)\Gamma_\alpha^a) \dot{\tilde{\partial}}_a,$$

and

$$(3.20) \quad \Gamma(\tilde{\rho}, Id_E) \left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta \right]_{(\rho, \eta)TE} = \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right), \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\beta \right) \right]_{TE}.$$

Let $(d\tilde{z}^\alpha, d\tilde{y}^b)$ be the natural dual (ρ, η) -base of natural (ρ, η) -base $(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a)$.

This is determined by the equations

$$\begin{cases} \langle d\tilde{z}^\alpha, \tilde{\partial}_\beta \rangle = \delta_\beta^\alpha, & \langle d\tilde{z}^\alpha, \dot{\tilde{\partial}}_a \rangle = 0, \\ \langle d\tilde{y}^a, \tilde{\partial}_\beta \rangle = 0, & \langle d\tilde{y}^a, \dot{\tilde{\partial}}_b \rangle = \delta_b^a. \end{cases}$$

We consider the problem of finding a base for the $\mathcal{F}(E)$ -module

$$(\Gamma((V(\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E), +, \cdot)$$

of the type

$$\delta\tilde{y}^a = \theta_\alpha^a d\tilde{z}^\alpha + \omega_b^a d\tilde{y}^b, \quad a \in \overline{1, n}$$

which satisfies the following conditions:

$$(3.21) \quad \langle \delta\tilde{y}^a, \dot{\tilde{\partial}}_a \rangle = 1 \wedge \langle \delta\tilde{y}^a, \tilde{\partial}_\alpha \rangle = 0.$$

We obtain the sections

$$(3.22) \quad \delta\tilde{y}^a = (\rho, \eta) \Gamma_\alpha^a d\tilde{z}^\alpha + d\tilde{y}^a, \quad a \in \overline{1, n}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base $(d\tilde{z}^\alpha, \delta\tilde{y}^a)$ will be called the *adapted dual (ρ, η) -base*.

4 The lift of a differentiable curve

We consider the following diagram:

$$(4.1) \quad \begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^v|$ and $((F, \nu, M), [\cdot, \cdot]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|$.

We admit that $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) .

Let

$$I \xrightarrow{c} M$$

be a differentiable curve.

We say that

$$(E|_{\text{Im}(\eta \circ h \circ c)}, \pi|_{\text{Im}(\eta \circ h \circ c)}, \text{Im}(\eta \circ h \circ c))$$

is a vector subbundle of the vector bundle (E, π, M) .

Definition 4.1 Let

$$(4.2) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & E|_{\text{Im}(\eta \circ h \circ c)} \\ t & \longmapsto & y^a(t) s_a(\eta \circ h \circ c(t)) \end{array}$$

be a differentiable curve.

If there exists $g \in \mathbf{Man}(E, F)$ such that the following conditions are satisfied:

1. $(g, h) \in \mathbf{B}^v((E, \pi, M), (F, \nu, N))$ and
2. $\rho \circ g \circ \dot{c}(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt} \frac{\partial}{\partial x^i}((\eta \circ h \circ c)(t))$, for any $t \in I$,

then we will say that \dot{c} is the (g, h) -lift of the differentiable curve c .

Remark 4.1 The condition 2 is equivalent with the following affirmation:

$$(4.3) \quad \rho_\alpha^i(\eta \circ h \circ c(t)) \cdot g_a^\alpha(h \circ c(t)) \cdot y^a(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt}, \quad i \in \overline{1, m}.$$

Definition 4.2 If

$$I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)}$$

is a differentiable (g, h) -lift of the differentiable curve c , then the section

$$(4.4) \quad \begin{array}{ccc} \text{Im}(\eta \circ h \circ c) & \xrightarrow{u(c, \dot{c})} & E|_{\text{Im}(\eta \circ h \circ c)} \\ \eta \circ h \circ c(t) & \longmapsto & \dot{c}(t) \end{array}$$

will be called the *canonical section associated to the couple (c, \dot{c})* .

Definition 4.3 If $(g, h) \in \mathbf{B}^v((E, \pi, M), (F, \nu, N))$ has the components

$$g_a^\alpha; a \in \overline{1, r}, \quad \alpha \in \overline{1, p}$$

such that for any vector local $(n + p)$ -chart (V, t_V) of (F, ν, N) there exists the real functions

$$V \xrightarrow{\tilde{g}_\alpha^a} \mathbb{R} ; a \in \overline{1, r}, \quad \alpha \in \overline{1, p}$$

such that

$$\tilde{g}_\alpha^b(\varkappa) \cdot g_a^\alpha(\varkappa) = \delta_a^b,$$

for any $\varkappa \in V$, then we will say that the \mathbf{B}^v -morphism (g, h) is *locally invertible*.

Remark 4.2 In particular, if $(Id_{TM}, Id_M, Id_M) = (\rho, \eta, h)$ and the \mathbf{B}^v morphism (g, Id_M) is locally invertible, then we have the differentiable (g, Id_M) -lift

$$(4.6) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \tilde{g}_j^i(c(t)) \frac{dc^j(t)}{dt} \frac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Moreover, if $g = Id_{TM}$, then we obtain the usual lift of tangent vectors

$$(4.6)' \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \frac{dc^i(t)}{dt} \frac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Definition 4.4 If

$$(4.7) \quad I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)}$$

is a differentiable (g, h) -lift of differentiable curve c , such that its components functions $(y^a, a \in \overline{1, n})$ are solutions for the differentiable system of equations:

$$(4.8) \quad \frac{du^a}{dt} + (\rho, \eta) \Gamma_\alpha^a \circ u(c, \dot{c}) \circ (\eta \circ h \circ c) \cdot g_b^\alpha \circ h \circ c \cdot u^b = 0,$$

then we will say that *the (g, h) -lift \dot{c} is parallel with respect to the (ρ, η) -connection $(\rho, \eta) \Gamma$.*

Remark 4.3 In particular, if $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$ and the \mathbf{B}^v morphism (g, Id_M) is locally invertible, then the differentiable (g, Id_{TM}) -lift

$$(4.9) \quad \begin{aligned} I &\xrightarrow{\dot{c}} TM \\ t &\longmapsto \left(\tilde{g}_j^i \circ c \cdot \frac{dc^j}{dt} \right) \frac{\partial}{\partial x^i} (c(t)), \end{aligned}$$

is parallel with respect to the connection Γ if the component functions

$$\left(\tilde{g}_j^i \circ c \cdot \frac{dc^j}{dt}, i \in \overline{1, n} \right)$$

are solutions for the differentiable system of equations

$$(4.10) \quad \frac{du^i}{dt} + \Gamma_k^i \circ u(c, \dot{c}) \circ c \cdot g_h^k \circ c \cdot u^h = 0,$$

namely

$$(4.10)' \quad \begin{aligned} &\frac{d}{dt} \left(\tilde{g}_j^i(c(t)) \cdot \frac{dc^j(t)}{dt} \right) \\ &+ \Gamma_k^i \left(\left(\tilde{g}_j^i(c(t)) \cdot \frac{dc^j(t)}{dt} \right) \cdot \frac{\partial}{\partial x^i} (c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0. \end{aligned}$$

Moreover, if $g = Id_{TM}$, then the usual lift of tangent vectors $(4.6)'$ is parallel with respect to the connection Γ if the component functions $\left(\frac{dc^j}{dt}, j \in \overline{1, n} \right)$ are solutions for the differentiable system of equations

$$(4.10)'' \quad \frac{du^i}{dt} + \Gamma_k^i \circ u(c, \dot{c}) \circ c \cdot u^k = 0,$$

namely

$$(4.10)''' \quad \frac{d}{dt} \left(\frac{dc^j(t)}{dt} \right) + \Gamma_k^i \left(\frac{dc^j(t)}{dt} \cdot \frac{\partial}{\partial x^i} (c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0.$$

5 Remarkable Mod-endomorphisms

In the following we consider the diagram:

$$\begin{array}{ccc} E & & (F, [,]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^V|$ and $\left((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

Definition 5.1 For any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ we define the application of Nijenhuis type

$$\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)^2 \xrightarrow{N_e} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

defined by

$$N_e(X, Y) = [eX, eY]_{\rho TE} + e^2[X, Y]_{\rho TE} - e[eX, Y]_{\rho TE} - e[X, eY]_{\rho TE},$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

5.1 Projectors

Definition 5.1.1 Any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the property

$$(5.1.1) \quad e^2 = e$$

will be called *projector*.

Example 5.1.1 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{V}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a &\longmapsto Y^a \dot{\tilde{\partial}}_a \end{aligned}$$

is a projector which will be called *the vertical projector*.

Remark 5.1.1 We have $\mathcal{V}(\tilde{\delta}_\alpha) = 0$ and $\mathcal{V}(\dot{\tilde{\partial}}_a) = \dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{V}(\dot{\tilde{\partial}}_\alpha) = (\rho, \eta) \Gamma_\alpha^a \dot{\tilde{\partial}}_a.$$

In addition, we obtain the equality

$$(5.1.2) \quad \Gamma((\rho, \eta)\Gamma, Id_E) \left(Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) = \mathcal{V} \left(Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \right),$$

for any $Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Theorem 5.1.1 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{V} of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the properties:

$$(5.1.3) \quad \begin{aligned} \mathcal{V}(\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)) &\subset \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ \mathcal{V}(X) = X &\iff X \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E) \end{aligned}$$

Example 5.1.2 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{H}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a &\longmapsto Z^\alpha \tilde{\delta}_\alpha \end{aligned}$$

is a projector which will be called *the horizontal projector*.

Remark 5.1.2 We have $\mathcal{H}(\tilde{\delta}_\alpha) = \tilde{\delta}_\alpha$ and $\mathcal{H}(\tilde{\partial}_a) = 0$. Therefore, we obtain $\mathcal{H}(\tilde{\delta}_\alpha) = \tilde{\delta}_\alpha$.

Theorem 5.1.2 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{H} of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the properties:

$$(5.1.4) \quad \begin{aligned} \mathcal{H}(\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)) &\subset \Gamma(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ \mathcal{H}(X) = X &\iff X \in \Gamma(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E). \end{aligned}$$

Corollary 5.1.1 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{H} of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the properties:

$$(5.1.5) \quad \begin{aligned} \mathcal{H}^2 &= \mathcal{H} \\ \text{Ker}(\mathcal{H}) &= (\Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot). \end{aligned}$$

Remark 5.1.3 For any $X \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ we obtain the unique decomposition

$$X = \mathcal{H}X + \mathcal{V}X.$$

Proposition 5.1.1 After some calculations we obtain

$$(5.1.6) \quad N_{\mathcal{V}}(X, Y) = \mathcal{V}[\mathcal{H}X, \mathcal{H}Y]_{(\rho, \eta)TE} = N_{\mathcal{H}}(X, Y),$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Corollary 5.1.2 The horizontal interior differential system

$$(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

is involutive if and only if $N_{\mathcal{V}} = 0$ or $N_{\mathcal{H}} = 0$.

5.2 The almost product structure

Definition 5.2.1 Any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the property

$$(5.2.1) \quad e^2 = Id$$

will be called the *almost product structure*.

Example 5.2.1 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{P}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \tilde{\partial}_a &\longmapsto Z^\alpha \tilde{\delta}_\alpha - Y^a \tilde{\partial}_a \end{aligned}$$

is an almost product structure.

Remark 5.2.1 The previous almost product structure has the properties:

$$(5.2.2) \quad \begin{aligned} \mathcal{P} &= (2\mathcal{H} - Id); \\ \mathcal{P} &= (Id - 2\mathcal{V}); \\ \mathcal{P} &= (\mathcal{H} - \mathcal{V}). \end{aligned}$$

Remark 5.2.2 We obtain that $\mathcal{P}(\tilde{\delta}_\alpha) = \tilde{\delta}_\alpha$ and $\mathcal{P}(\dot{\tilde{\partial}}_a) = -\dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{P}(\tilde{\partial}_\alpha) = \tilde{\delta}_\alpha - \rho \Gamma_\alpha^a \dot{\tilde{\partial}}_a.$$

Theorem 5.2.1 *A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{P} of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the following property:*

$$(5.2.3) \quad \mathcal{P}(X) = -X \iff X \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E).$$

Proposition 5.2.1 *After some calculations, we obtain*

$$N_{\mathcal{P}}(X, Y) = 4\mathcal{V}[\mathcal{H}X, \mathcal{H}Y],$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Corollary 5.2.1 *The horizontal interior differential system $(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is involutive if and only if $N_{\mathcal{P}} = 0$.*

5.3 The almost tangent structure

Definition 5.3.1 Any **Mod**-endomorphism e of $(\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E))$ with the property

$$(5.3.1) \quad e^2 = 0$$

will be called the *almost tangent structure*.

Example 5.3.1 If $(E, \pi, M) = (F, \nu, N)$, $g \in \mathbf{Man}(E, E)$ such that (g, h) is a locally invertible **B^v**-morphism, then the **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{J}_{(g, h)}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^a \tilde{\delta}_a + Y^b \dot{\tilde{\partial}}_b &\longmapsto (\tilde{g}_a^b \circ h \circ \pi) Z^a \dot{\tilde{\partial}}_b \end{aligned}$$

is an almost tangent structure which will be called the *almost tangent structure associated to the **B^v**-morphism (g, h)* . (See: Definition 4.3)

Remark 5.3.1 We obtain that

$$\mathcal{J}_{(g, h)}(\tilde{\delta}_a) = \mathcal{J}_{(g, h)}(\tilde{\partial}_a) = (\tilde{g}_a^b \circ h \circ \pi) \dot{\tilde{\partial}}_b \text{ and } \mathcal{J}_{(g, h)}(\dot{\tilde{\partial}}_b) = 0.$$

and we have the following properties:

$$(5.3.2) \quad \begin{aligned} \mathcal{J}_{(g, h)} \circ \mathcal{P} &= \mathcal{J}_{(g, h)}; \\ \mathcal{P} \circ \mathcal{J}_{(g, h)} &= -\mathcal{J}_{(g, h)}; \\ \mathcal{J}_{(g, h)} \circ \mathcal{H} &= \mathcal{J}_{(g, h)}; \\ \mathcal{H} \circ \mathcal{J}_{(g, h)} &= 0; \\ \mathcal{J}_{(g, h)} \circ \mathcal{V} &= 0; \\ \mathcal{V} \circ \mathcal{J}_{(g, h)} &= \mathcal{J}_{(g, h)}; \\ N_{\mathcal{J}_{(g, h)}} &= 0. \end{aligned}$$

6 Tensor d -fields. Distinguished linear (ρ, η) -connections

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^V|$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid.

Let

$$(\mathcal{T}_{q,s}^{p,r}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$$

be the $\mathcal{F}(E)$ -module of tensor fields by $(\frac{p,r}{q,s})$ -type from the generalized tangent bundle

$$(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E) \oplus (V(\rho, \eta)TE, (\rho, \eta)\tau_E, E).$$

An arbitrarily tensor field T is written as

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}.$$

Let

$$(\mathcal{T}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

If $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ and $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$, then the components of product tensor field $T_1 \otimes T_2$ are the products of local components of T_1 and T_2 .

Therefore, we obtain $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Let $\mathcal{DT}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ be the family of tensor fields

$$T \in \mathcal{T}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0}((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \text{ and } T_2 \in \mathcal{T}_{0,s}^{0,r}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

such that $T = T_1 + T_2$.

The $\mathcal{F}(E)$ -module $(\mathcal{DT}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$ will be called the *module of distinguished tensor fields* or the *module of tensor d -fields*.

Remark 5.1 The elements of

$$\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

respectively

$$\Gamma(((\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E)$$

are tensor d -fields.

Definition 6.1 Let (E, π, M) be a vector bundle endowed with a (ρ, η) -connection $(\rho, \eta) \Gamma$ and let

$$(6.1) \quad (X, T) \xrightarrow{(\rho, \eta) D} (\rho, \eta) D_X T$$

be a covariant (ρ, η) -derivative for the tensor algebra of the generalized tangent bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

which preserves the horizontal and vertical IDS by parallelism.

The real local functions

$$((\rho, \eta) H_{\beta\gamma}^\alpha, (\rho, \eta) H_{b\gamma}^a, (\rho, \eta) V_{\beta c}^\alpha, (\rho, \eta) V_{bc}^a)$$

defined by the following equalities:

$$(6.2) \quad \begin{aligned} (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta &= (\rho, \eta) H_{\beta\gamma}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\partial}_b &= (\rho, \eta) H_{b\gamma}^a \tilde{\partial}_a \\ (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\delta}_\beta &= (\rho, \eta) V_{\beta c}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\partial}_c} \tilde{\partial}_b &= (\rho, \eta) V_{bc}^a \tilde{\partial}_a \end{aligned}$$

are the components of a linear (ρ, η) -connection

$$((\rho, \eta) H, (\rho, \eta) V)$$

for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ which will be called the *distinguished linear (ρ, η) -connection*.

If $h = Id_M$, then the distinguished linear (Id_{TM}, Id_M) -connection will be called the *distinguished linear connection*.

The components of a distinguished linear connection (H, V) will be denoted

$$(H_{jk}^i, H_{bk}^a, V_{jc}^i, V_{bc}^a).$$

Theorem 6.1 *If $((\rho, \eta)H, (\rho, \eta)V)$ is a distinguished linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, then its components satisfy the change relations:*

$$(6.3) \quad \begin{aligned} (\rho, \eta) H_{\beta\gamma'}^{\alpha'} &= \Lambda_{\alpha'}^{\alpha} \circ h \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\gamma \right) \left(\Lambda_{\beta'}^{\alpha} \circ h \circ \pi \right) + \right. \\ &\quad \left. + (\rho, \eta) H_{\beta\gamma}^{\alpha} \cdot \Lambda_{\beta'}^{\beta} \circ h \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ h \circ \pi, \\ (\rho, \eta) H_{b\gamma'}^a &= M_a^a \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\gamma \right) (M_b^a \circ \pi) + \right. \\ &\quad \left. + (\rho, \eta) H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma'}^{\gamma} \circ h \circ \pi, \\ (\rho, \eta) V_{\beta c'}^{\alpha'} &= \Lambda_{\alpha'}^{\alpha} \circ h \circ \pi \cdot (\rho, \eta) V_{\beta c}^{\alpha} \cdot \Lambda_{\beta'}^{\beta} \circ h \circ \pi \cdot M_c^c \circ \pi, \\ (\rho, \eta) V_{bc'}^a &= M_a^a \circ \pi \cdot (\rho, \eta) V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned}$$

The components of a distinguished linear connection (H, V) verify the change relations:

$$\begin{aligned}
(6.3') \quad H_{jk'}^i &= \frac{\partial x^i}{\partial x^k} \circ \pi \cdot \left[\frac{\delta}{\delta x^k} \left(\frac{\partial x^i}{\partial x^j} \circ \pi \right) + H_{jk}^i \cdot \frac{\partial x^j}{\partial x^k} \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\
H_{bk'}^a &= M_a^a \circ \pi \cdot \left[\frac{\delta}{\delta x^k} (M_b^a \circ \pi) + H_{bk}^a \cdot M_b^b \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\
V_{jc'}^i &= \frac{\partial x^i}{\partial x^j} \circ \pi \cdot V_{jc}^i \frac{\partial x^j}{\partial x^c} \circ \pi \cdot M_c^c \circ \pi, \\
V_{bc'}^a &= M_a^a \circ \pi \cdot V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi.
\end{aligned}$$

Example 6.1 If (E, π, M) is a vector bundle endowed with the (ρ, η) -connection $(\rho, \eta) \Gamma$, then the local real functions

$$\left(\frac{\partial(\rho, \eta) \Gamma_\gamma^a}{\partial y^b}, \frac{\partial(\rho, \eta) \Gamma_\gamma^a}{\partial y^b}, 0, 0 \right)$$

are the components of a distinguished linear (ρ, η) -connection for $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, which will be called the *Berwald linear (ρ, η) -connection*.

The Berwald linear (Id_{TM}, Id_M) -connection will be called the *Berwald linear connection*.

Theorem 6.2 If the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ is endowed with a distinguished linear (ρ, η) -connection $((\rho, \eta) H, (\rho, \eta) V)$, then, for any

$$X = \tilde{Z}^\alpha \tilde{\delta}_\alpha + Y^a \tilde{\partial}_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr}((\rho, \eta) TE, (\rho, \eta) \tau_E, E),$$

we obtain the formula:

$$\begin{aligned}
(\rho, \eta) DX &\left(T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\
&\quad \left. \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} \right) = \\
&= \tilde{Z}^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \\
&\quad \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} + Y^c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \\
&\quad \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s},
\end{aligned}$$

where

$$\begin{aligned}
T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ (\rho, \eta) H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\
&- (\rho, \eta) H_{\beta_1 \gamma}^\beta T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^\beta T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ (\rho, \eta) H_{a_1 \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) H_{a_r \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&- (\rho, \eta) H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\partial}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ (\rho, \eta) V_{\alpha c}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_{\alpha c}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} - \\
&- (\rho, \eta) V_{\beta_1 c}^{\beta_1} T_{\beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q c}^{\beta_q} T_{\beta_1 \dots \beta_{q-1} b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ (\rho, \eta) V_{ac}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) V_{ac}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} - \\
&- (\rho, \eta) V_{b_1 c}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

Definition 6.2 We assume that $(E, \pi, M) = (F, \nu, N)$.

If $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) and

$$\left((\rho, \eta) H_{bc}^a, (\rho, \eta) \tilde{H}_{bc}^a, (\rho, \eta) V_{bc}^a, (\rho, \eta) \tilde{V}_{bc}^a \right)$$

are the components of a distinguished linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ such that

$$(\rho, \eta) H_{bc}^a = (\rho, \eta) \tilde{H}_{bc}^a \text{ and } (\rho, \eta) V_{bc}^a = (\rho, \eta) \tilde{V}_{bc}^a,$$

then we will say that the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ is endowed with a normal distinguished linear (ρ, η) -connection on components $((\rho, \eta) H_{bc}^a, (\rho, \eta) V_{bc}^a)$.

The components of a normal distinguished linear (Id_{TM}, Id_M) -connection (H, V) will be denoted (H_{jk}^i, V_{jk}^i) .

7 Mechanical systems

We consider the following diagram:

$$(7.1) \quad \begin{array}{ccc} E & & (E, [\cdot, \cdot]_{E,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{h} & M \end{array}$$

where $((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta))$ is a generalized Lie algebroid.

Definition 7.1 A triple

$$(7.2) \quad ((E, \pi, M), F_e, (\rho, \eta) \Gamma),$$

where

$$(7.3) \quad F_e = F^a \frac{\partial}{\partial y^a} \in \Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

is an external force and $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) , will be called the *mechanical (ρ, η) -system*.

Definition 7.2 A smooth *Lagrange fundamental function* on the vector bundle (E, π, M) is a mapping $E \xrightarrow{L} \mathbb{R}$ which satisfies the following conditions:

1. $L \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $L \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M) .

Let L be a Lagrangian defined on the total space of the vector bundle (E, π, M) .

If (U, s_U) is a local vector $(m+r)$ -chart for (E, π, M) , then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(7.5) \quad \begin{aligned} L_i &\stackrel{put}{=} \frac{\partial L}{\partial x^i} \stackrel{put}{=} \frac{\partial}{\partial x^i}(L) & L_{ib} &\stackrel{put}{=} \frac{\partial^2 L}{\partial x^i \partial y^b} \stackrel{put}{=} \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial y^b}(L) \right) \\ L_a &\stackrel{put}{=} \frac{\partial L}{\partial y^a} \stackrel{put}{=} \frac{\partial}{\partial y^a}(L) & L_{ab} &\stackrel{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \stackrel{put}{=} \frac{\partial}{\partial y^a} \left(\frac{\partial}{\partial y^b}(L) \right). \end{aligned}$$

Definition 7.3 If for any vector local $m+r$ -chart (U, s_U) of (E, π, M) , we have:

$$(7.6) \quad \text{rank} \|L_{ab}(u_x)\| = r,$$

for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$, then we will say that *the Lagrangian L is regular*.

Proposition 7.1 *If the Lagrangian L is regular, then for any vector local $m+r$ -chart (U, s_U) of (E, π, M) , we obtain the real functions \tilde{L}^{ab} locally defined by*

$$(7.8) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{L}^{ab}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{L}^{ab}(u_x) \end{array},$$

where $\|\tilde{L}^{ab}(u_x)\| = \|L_{ab}(u_x)\|^{-1}$, for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 7.4 A smooth *Finsler fundamental function* on the vector bundle (E, π, M) is a mapping

$$E \xrightarrow{F} \mathbb{R}_+$$

which satisfies the following conditions:

1. $F \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $F \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M) ;
3. F is positively 1-homogenous on the fibres of vector bundle (E, π, M) ;
4. For any vector local $m+r$ -chart (U, s_U) of (E, π, M) , the hessian:

$$(7.9) \quad \|F_{ab}^2(u_x)\|$$

is positively define for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 7.5 If L (respectively F) is a smooth Lagrange (respectively Finsler function), then the triple

$$((E, \pi, M), F_e, L) \quad (\text{respectively } ((E, \pi, M), F_e, F))$$

where $F_e = F^a \frac{\partial}{\partial \tilde{y}^a} \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is an external force, is called *Lagrange mechanical (ρ, η) -system* and *Finsler mechanical (ρ, η) -system*, respectively.

Any Lagrange mechanical (Id_{TM}, Id_M) -system and any Finsler mechanical (Id_{TM}, Id_M) -system will be called *Lagrange mechanical system* and *Finsler mechanical system*, respectively.

8 (ρ, η) -semisprays and (ρ, η) -sprays for mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, (\rho, \eta)\Gamma)$ be an arbitrary mechanical (ρ, η) -system.

Definition 8.1 The vertical section $\mathbb{C} = y^a \dot{\tilde{\partial}}_a$ will be called the *Liouville section*.

A section $S \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ will be called (ρ, η) -semispray if there exists an almost tangent structure e such that $e(S) = \mathbb{C}$.

Let $g \in \mathbf{Man}(E, E)$ be such that (g, h) is a locally invertible \mathbf{B}^v -morphism of (E, π, M) source and target.

Theorem 8.1 *The section*

$$(8.2) \quad S = (g_b^a \circ h \circ \pi) y^b \tilde{\partial}_a - 2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a$$

is a (ρ, η) -semispray such that the real local functions G^a , $a \in \overline{1, n}$, satisfy the following conditions

$$(8.3) \quad \begin{aligned} (\rho, \eta) \Gamma_c^a &= (\tilde{g}_c^b \circ h \circ \pi) \frac{\partial(G^a - \frac{1}{4} F^a)}{\partial y^b} \\ &\quad - \frac{1}{2} (g_e^d \circ h \circ \pi) y^e L_{dc}^f (\tilde{g}_f^a \circ h \circ \pi) \\ &\quad + \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad - \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} \end{aligned}$$

In addition, we remark that the local real functions

$$(8.4) \quad \begin{aligned} (\rho, \eta) \dot{\Gamma}_c^a &= (\tilde{g}_c^b \circ h \circ \pi) \frac{\partial G^a}{\partial y^b} \\ &\quad - \frac{1}{2} (g_e^d \circ h \circ \pi) y^e L_{dc}^b (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad + \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad - \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} \end{aligned}$$

are the components of a (ρ, η) -connection $(\rho, \eta) \dot{\Gamma}$ for the vector bundle (E, π, M) .

The (ρ, η) -semispray S will be called the *canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta)\Gamma)$ and from locally invertible \mathbf{B}^v -morphism (g, h)* .

Proof. We consider the **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathbb{P}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ X &\longmapsto \mathcal{J}_{(g, h)}[S, X]_{(\rho, \eta)TE} - [S, \mathcal{J}_{(g, h)}X]_{(\rho, \eta)TE}. \end{aligned}$$

Let $X = Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a$ be an arbitrary section. Since

$$\begin{aligned} [S, X]_{(\rho, \eta)TE} &= \left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta)TE} + \left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} \\ &\quad - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta)TE} - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} \end{aligned}$$

and

$$\begin{aligned}
\left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta)TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^c}{\partial x^i} \tilde{\partial}_c \\
&\quad - Z^b \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e \tilde{\partial}_c \\
&\quad + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c \tilde{\partial}_c, \\
\left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Y^c}{\partial x^i} \dot{\tilde{\partial}}_c \\
&\quad - Y^b g_b^c \dot{\tilde{\partial}}_c, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta)TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} \tilde{\partial}_c \\
&\quad - 2 Z^b \rho_b^j \circ h \circ \pi \frac{\partial (G^c - \frac{1}{4} F^c)}{\partial x^j} \dot{\tilde{\partial}}_c, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Y^c}{\partial y^a} \dot{\tilde{\partial}}_c - 2 Y^b \frac{\partial \left(G^c - \frac{1}{4} F^c \right)}{\partial y^b} \dot{\tilde{\partial}}_c,
\end{aligned}$$

it results that

$$\begin{aligned}
\mathcal{J}_{(g, h)} [S, X]_{(\rho, \eta)TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^c}{\partial x^i} \left(\tilde{g}_c^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d \\
&\quad - Z^b \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_c^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d \\
&\quad + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c \left(\tilde{g}_c^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d - Y^d \dot{\tilde{\partial}}_d \\
&\quad - 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} \left(\tilde{g}_c^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d.
\end{aligned}
\tag{P_1}$$

Since

$$\begin{aligned}
[S, \mathcal{J}_{(g, h)} X]_{(\rho, \eta)TE} &= \left[(g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho, \eta)TE} \\
&\quad - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho, \eta)TE}
\end{aligned}$$

and

$$\begin{aligned}
\left[(g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho, \eta)TE} &= -Z^d \dot{\tilde{\partial}}_d + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^b}{\partial x^i} \left(\tilde{g}_b^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d \\
&\quad - (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho, \eta)TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^a} \left(\tilde{g}_b^d \circ h \circ \pi \right) \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 \left(G^d - \frac{1}{4} F^d \right)}{\partial y^c} \dot{\tilde{\partial}}_d
\end{aligned}$$

it results that

$$\begin{aligned}
(P_2) \quad [S, \mathcal{J}_{(g,h)} X]_{(\rho,\eta)TE} &= -Z^d \tilde{\partial}_d + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^b}{\partial x^i} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\
&\quad - 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^a} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d.
\end{aligned}$$

Using equalities (P_1) and (P_2) , we obtain:

$$\begin{aligned}
\mathbb{P} \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) &= Z^a \tilde{\partial}_a - Y^d \dot{\tilde{\partial}}_d + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - Z^b \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d
\end{aligned}$$

After some calculations, it results that \mathbb{P} is an almost product structure.

Using the equalities (5.1.2) and (5.2.2) it results that

$$\mathbb{P} \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) = (Id - 2(\rho, \eta) \Gamma) \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right),$$

for any $Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ and we obtain

$$\begin{aligned}
(\rho, \eta) \Gamma \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) &= Y^d \dot{\tilde{\partial}}_d - \frac{1}{2} (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + \frac{1}{2} Z^b \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - \frac{1}{2} (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\
&\quad + Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d.
\end{aligned}$$

Since

$$(\rho, \eta) \Gamma \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) = (Y^d + (\rho, \eta) \Gamma_b^d Z^b) \dot{\tilde{\partial}}_d$$

it results the relations (8.3). In addition, since

$$(\rho, \eta) \mathring{\Gamma}_c^a = (\rho, \eta) \Gamma_c^a + \frac{1}{4} \tilde{g}_c^d \circ h \circ \pi \frac{\partial F^a}{\partial y^d}$$

and

$$\begin{aligned}
(\rho, \eta) \mathring{\Gamma}_c^{a'} &= (\rho, \eta) \Gamma_c^{a'} + \frac{1}{2} \tilde{g}_c^{b'} \circ h \circ \pi \frac{\partial F^{a'}}{\partial y^{b'}} \\
&= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + (\rho, \eta) \Gamma_c^a \right) M_c^c \circ h \circ \pi \\
&\quad + M_a^{a'} \circ \pi \left(\frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \right) M_c^c \circ h \circ \pi \\
&= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + \left((\rho, \eta) \Gamma_c^a + \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \right) \right) M_c^c \circ h \circ \pi \\
&= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + (\rho, \eta) \mathring{\Gamma}_c^a \right) M_c^c \circ h \circ \pi
\end{aligned}$$

it results the conclusion of the theorem. q.e.d.

Remark 8.1 If $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to connection Γ which is not the same canonical semispray presented by I. Bucataru and R. Miron in [5].

In particular, if $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e = 0$, then we obtain the classical canonical semispray associated to connection Γ .

Using *Theorem 8.1*, we obtain the following:

Theorem 8.2 *The following properties hold good:*

1° Since $\mathring{\tilde{\delta}}_c = \tilde{\delta}_c - (\rho, \eta) \mathring{\Gamma}_c^a \dot{\tilde{\delta}}_a$, $c \in \overline{1, r}$, it results that

$$(8.5) \quad \mathring{\tilde{\delta}}_c = \tilde{\delta}_c - \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \dot{\tilde{\delta}}_a, \quad c \in \overline{1, r}.$$

2° Since $\mathring{\delta} \tilde{y}^a = (\rho, \eta) \mathring{\Gamma}_c^a d\tilde{z}^c + d\tilde{y}^a$, it results that

$$(8.6) \quad \mathring{\delta} \tilde{y}^a = \delta \tilde{y}^a + \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \frac{\partial F^a}{\partial y^b} d\tilde{z}^c, \quad a \in \overline{1, r}.$$

Theorem 8.3 *The real local functions*

$$(8.7) \quad \left(\frac{\partial(\rho, \eta) \Gamma_c^a}{\partial y^b}, \frac{\partial(\rho, \eta) \Gamma_c^a}{\partial y^b}, 0, 0 \right), \quad a, b, c \in \overline{1, r},$$

and

$$(8.7') \quad \left(\frac{\partial(\rho, \eta) \mathring{\Gamma}_c^a}{\partial y^b}, \frac{\partial(\rho, \eta) \mathring{\Gamma}_c^a}{\partial y^b}, 0, 0 \right), \quad a, b, c \in \overline{1, r},$$

respectively, are the coefficients to a normal Berwald linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Theorem 8.4 *The tensor of integrability of the (ρ, η) -connection $(\rho, \eta) \mathring{\Gamma}$ is as follows:*

$$\begin{aligned}
(\rho, \eta, h) \mathring{\mathbb{R}}_{cd}^a &= (\rho, \eta, h) \mathbb{R}_{cd}^a + \frac{1}{4} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \Big|_c - \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \Big|_d \right) \\
(8.8) \quad &+ \frac{1}{16} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^b}{\partial y^e} \tilde{g}_c^f \circ h \circ \pi \frac{\partial^2 F^a}{\partial y^b \partial y^f} - \tilde{g}_c^f \circ h \circ \pi \frac{\partial F^b}{\partial y^f} \tilde{g}_d^e \circ h \circ \pi \frac{\partial^2 F^a}{\partial y^b \partial y^e} \right) \\
&+ \frac{1}{4} \left(L_{cd}^f \circ h \circ \pi \right) \left(\tilde{g}_f^e \circ h \circ \pi \right) \frac{\partial F^a}{\partial y^e},
\end{aligned}$$

where $|_c$ is the h -covariant derivation with respect to the normal Berwald linear (ρ, η) -connection (8.7).

Proof. Since

$$\begin{aligned} (\rho, \eta, h) \mathring{\mathbb{R}}_{cd}^a &= \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_c \right) \left((\rho, \eta) \mathring{\Gamma}_d^a \right) - \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_d \right) \left((\rho, \eta) \mathring{\Gamma}_c^a \right) \\ &\quad + L_{cd}^e \circ h \circ (h \circ \pi) (\rho, \eta) \mathring{\Gamma}_e^a, \end{aligned}$$

and

$$\begin{aligned} \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_c \right) \left((\rho, \eta) \mathring{\Gamma}_d^a \right) &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_c \right) ((\rho, \eta) \Gamma_d^a) \\ &\quad + \frac{1}{4} \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_c \right) \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right) \\ &\quad - \frac{1}{4} \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} ((\rho, \eta) \Gamma_d^a) \\ &\quad - \frac{1}{16} \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right), \\ \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_d \right) \left((\rho, \eta) \mathring{\Gamma}_c^a \right) &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_d \right) ((\rho, \eta) \Gamma_c^a) \\ &\quad + \frac{1}{4} \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_d \right) \left(\tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right) \\ &\quad - \frac{1}{4} \tilde{g}_d^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} ((\rho, \eta) \Gamma_c^a) \\ &\quad - \frac{1}{16} \tilde{g}_d^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} \left(\tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right), \\ L_{cd}^e \circ h \circ \pi \cdot (\rho, \eta) \mathring{\Gamma}_e^a &= L_{cd}^e \circ h \circ \pi \cdot (\rho, \eta) \Gamma_e^a \\ &\quad + L_{cd}^e \circ h \circ \pi \cdot \left(\tilde{g}_e^f \circ h \circ \pi \frac{\partial F^a}{\partial y^f} \right) \end{aligned}$$

it results the conclusion of the theorem.

q.e.d.

Proposition 8.1 *If S is the canonical (ρ, η) -semispray associated to the mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from \mathbf{B}^v -morphism (g, h) , then*

$$(8.9) \quad 2G^{a'} = 2G^a M_a^{a'} \circ h \circ \pi - (g_b^a \circ h \circ \pi) y^b (\rho_a^i \circ h \circ \pi) \frac{\partial y^{a'}}{\partial x^i}.$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial M_a^{a'} \circ \pi}{\partial x^i} y^a & M_a^{a'} \circ \pi \end{array} \right\| = \left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial y^{a'}}{\partial x^i} & M_a^{a'} \circ \pi \end{array} \right\|$$

and

$$\left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial y^{a'}}{\partial x^i} & M_a^{a'} \circ \pi \end{array} \right\| \cdot \left(\begin{array}{c} (g_b^a \circ h \circ \pi) y^b \\ -2 \left(G^a - \frac{1}{4} F^a \right) \end{array} \right) = \left(\begin{array}{c} (g_b^{a'} \circ h \circ \pi) y^{b'} \\ -2 \left(G^{a'} - \frac{1}{4} F^{a'} \right) \end{array} \right),$$

the conclusion results immediately.

In the following, we consider a differentiable curve $I \xrightarrow{c} M$ and its (g, h) -lift $I \xrightarrow{\dot{c}} E$. *q.e.d.*

Definition 8.3 If it verifies the following equality:

$$(8.10) \quad \frac{d\dot{c}(t)}{dt} = \Gamma(\tilde{\rho}, Id_E) S(\dot{c}(t)),$$

then we say that *the curve \dot{c} is an integral curve of the (ρ, η) -semispray S of the mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$,*

Theorem 8.5 *All (g, h) -lifts solutions of the equations:*

$$(8.11) \quad \frac{dy^a(t)}{dt} + 2G^a \circ u(c, \dot{c})(x(t)) = \frac{1}{2} F^a \circ u(c, \dot{c})(x(t)), \quad a \in \overline{1, r},$$

where $x(t) = (\eta \circ h \circ c)(t)$, are integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^v -morphism (g, h) .

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt} = \Gamma(\tilde{\rho}, Id_E) S(\dot{c}(t))$$

is equivalent to

$$\begin{aligned} & \frac{d}{dt}((\eta \circ h \circ c)^i(t), y^a(t)) \\ &= \left(\rho_a^i \circ \eta \circ h \circ c(t) g_b^a \circ h \circ c(t) y^b(t), -2 \left(G^a - \frac{1}{4} F^a \right) ((\eta \circ h \circ c)^i(t), y^a(t)) \right), \end{aligned}$$

it results

$$\begin{aligned} & \frac{dy^a(t)}{dt} + 2G^a(x^i(t), y^a(t)) = \frac{1}{2} F^a(x^i(t), y^a(t)), \quad a \in \overline{1, n}, \\ & \frac{dx^i(t)}{dt} = \rho_a^i \circ \eta \circ h \circ c(t) g_b^a \circ h \circ c(t) y^b(t), \end{aligned}$$

where $x^i(t) = (\eta \circ h \circ c)^i(t)$.

q.e.d.

Definition 8.4 If S is a (ρ, η) -semispray, then the vector field

$$(8.12) \quad [\mathbb{C}, S]_{(\rho, \eta)TE} - S$$

will be called the *derivation of (ρ, η) -semispray S* .

The (ρ, η) -semispray S will be called (ρ, η) -*spray* if the following conditions are verified:

1. $S \circ 0 \in C^1$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ, η) -semispray S will be called *quadratic (ρ, η) -spray* if there are verified the following conditions:

1. $S \circ 0 \in C^2$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho, \eta) = (id_{TM}, Id_M)$ and $(g, h) = (Id_E, Id_M)$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 8.6 *If S is the canonical (ρ, η) -spray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) , then*

$$\begin{aligned}
 2 \left(G^a - \frac{1}{4} F^a \right) &= (\rho, \eta) \Gamma_c^a \left(g_f^c \circ h \circ \pi \right) y^f \\
 &+ \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \\
 &- \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \\
 &+ \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e \left(\rho_b^i \circ h \circ \pi \right) \frac{\partial (\tilde{g}_e^a \circ h \circ \pi)}{\partial x^i} \left(g_f^c \circ h \circ \pi \right) y^f
 \end{aligned}
 \tag{8.13}$$

We obtain the spray

$$\begin{aligned}
 S &= (g_b^a \circ h \circ \pi) y^b \dot{\tilde{\partial}}_a + (\rho, \eta) \Gamma_c^a \left(g_f^c \circ h \circ \pi \right) y^f \dot{\tilde{\partial}}_a \\
 &+ \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \dot{\tilde{\partial}}_a \\
 &- \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \dot{\tilde{\partial}}_a \\
 &+ \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e \left(\rho_b^i \circ h \circ \pi \right) \frac{\partial (\tilde{g}_e^a \circ h \circ \pi)}{\partial x^i} \left(g_f^c \circ h \circ \pi \right) y^f \dot{\tilde{\partial}}_a
 \end{aligned}
 \tag{8.14}$$

This spray will be called the canonical (ρ, η) -spray associated to mechanical system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .

In particular, if $(\rho, \eta) = (id_{TM}, Id_M)$ and $(g, h) = (Id_E, Id_M)$, then we get the canonical spray associated to connection Γ which is similar with the classical canonical spray associated to connection Γ .

Proof. Since

$$\begin{aligned}
 [\mathbb{C}, S]_{(\rho, \eta)TE} &= \left[y^a \dot{\tilde{\partial}}_a, (g_e^b \circ h \circ \pi \cdot y^e) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} - 2 \left[y^a \dot{\tilde{\partial}}_a, (G^b - \frac{1}{4} F^b) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE}, \\
 \left[y^a \dot{\tilde{\partial}}_a, (g_e^b \circ h \circ \pi \cdot y^e) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} &= \left(g_e^b \circ h \circ \pi \cdot y^e \right) \dot{\tilde{\partial}}_b
 \end{aligned}$$

and

$$\left[y^a \dot{\tilde{\partial}}_a, (G^b - \frac{1}{4} F^b) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} = y^a \frac{\partial (G^b - \frac{1}{4} F^b)}{\partial y^a} \dot{\tilde{\partial}}_b - \left(G^b - \frac{1}{4} F^b \right) \dot{\tilde{\partial}}_b$$

it results that

$$(S_1) \quad [\mathbb{C}, S]_{(\rho, \eta)TE} - S = 2 \left(-y^f \frac{\partial (G^a - \frac{1}{4} F^a)}{y^f} + 2 \left(G^a - \frac{1}{4} F^a \right) \right) \dot{\tilde{\partial}}_a$$

Using equality (8.3), it results that

$$\begin{aligned}
 \frac{\partial (G^a - \frac{1}{4} F^a)}{y^f} &= (\rho, \eta) \Gamma_c^a \left(g_f^c \circ h \circ \pi \right) \\
 &+ \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) \\
 &- \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) \\
 &+ \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e \left(\rho_b^i \circ h \circ \pi \right) \frac{\partial (\tilde{g}_e^a \circ h \circ \pi)}{\partial x^i} \left(g_f^c \circ h \circ \pi \right)
 \end{aligned}
 \tag{S_2}$$

Using equalities (S_1) and (S_2) , it results the conclusion of the theorem. *q.e.d.*

Theorem 8.7 *All (g, h) -lifts solutions of the following system of equations:*

$$(8.15) \quad \begin{aligned} & \frac{dy^a}{dt} + (\rho, \eta) \Gamma_c^a (g_f^c \circ h \circ \pi) y^f \\ & + \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \\ & - \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \\ & + \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial (\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} (g_f^c \circ h \circ \pi) y^f = 0, \end{aligned}$$

are the integral curves of canonical (ρ, η) -spray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^v -morphism (g, h) .

9 A Lagrangian formalism for Lagrange mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, L)$ be an arbitrarily Lagrange mechanical (ρ, η) -system.

Let $(d\tilde{z}^a, d\tilde{y}^a)$ be the natural dual (ρ, η) -base of the natural (ρ, η) -base $\left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a \right)$.

It is very important to remark that the 1-forms $d\tilde{z}^a, d\tilde{y}^a$, $a \in \overline{1, p}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

$$(d\tilde{z}^a) \neq d^{(\rho, \eta)TE}(\tilde{z}^a) = 0,$$

where $d^{(\rho, \eta)TE}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}(E)$ -algebra

$$(\wedge((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot, \wedge).$$

Let L be a regular Lagrangian and let (g, h) be a locally invertible \mathbf{B}^v -morphism of (E, π, M) source and (E, π, M) target.

Definition 9.1 The 1-form

$$(9.1) \quad \theta_L = (\tilde{g}_a^e \circ h \circ \pi \cdot L_e) d\tilde{z}^a$$

will be called the 1-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible \mathbf{B}^v -morphism (g, h) .

We obtain easily:

$$(9.2) \quad \theta_L \left(\tilde{\partial}_a \right) = \tilde{g}_b^e \circ h \circ \pi \cdot L_e, \quad \theta_L \left(\dot{\tilde{\partial}}_b \right) = 0.$$

Definition 9.2 The 2-form

$$\omega_L = d^{(\rho, \eta)TE} \theta_L$$

will be called the 2-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible \mathbf{B}^v -morphism (g, h) .

By the definition of $d^{(\rho,\eta)TE}$, we obtain:

$$(9.3) \quad \begin{aligned} \omega_L(U, V) &= \Gamma(\tilde{\rho}, Id_E)(U)(\theta_L(V)) \\ &\quad - \Gamma(\tilde{\rho}, Id_E)(V)(\theta_L(U)) - \theta_L([U, V]_{(\rho,\eta)TE}), \end{aligned}$$

for any $U, V \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Definition 9.3 The real function

$$(9.4) \quad \mathcal{E}_L = (g_e^a \circ h \circ \pi) y^e L_a - L$$

will be called the *energy of regular Lagrangian L*.

Theorem 9.1 The equation

$$(9.5) \quad i_S(\omega_L) = -d^{(\rho,\eta)TE}(\mathcal{E}_L), \quad S \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E),$$

has an unique solution $S_L(g, h)$ of the type:

$$(9.6) \quad (g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a - 2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a,$$

where

$$(9.7) \quad -2 \left(G^a - \frac{1}{4} F^a \right) = E_b(L, g, h) \tilde{L}^{be} (g_e^a \circ h \circ \pi)$$

and

$$(9.8) \quad \begin{aligned} E_b(L, g, h) &= (\rho_b^i \circ h \circ \pi) L_i \\ &\quad - (\rho_b^i \circ h \circ \pi) \frac{\partial((g_e^a \circ h \circ \pi) y^e L_a)}{\partial x^i} \\ &\quad - \left(g_f^d \circ h \circ \pi \right) y^f (\rho_d^i \circ h \circ \pi) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial x^i} \\ &\quad + \left(g_f^d \circ h \circ \pi \right) y^f (\rho_b^i \circ h \circ \pi) \frac{\partial((\tilde{g}_d^e \circ h \circ \pi) L_e)}{\partial x^i} \\ &\quad + \left(g_f^d \circ h \circ \pi \right) y^f (L_{ab}^c \circ h \circ \pi) (\tilde{g}_c^e \circ h \circ \pi) L_e \end{aligned}$$

$S_L(g, h)$ will be called the *canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .*

Proof. We obtain that

$$i_S(\omega_L) = -d^{(\rho,\eta)TE}(\mathcal{E}_L)$$

if and only if

$$\omega_L(S, X) = -\Gamma(\tilde{\rho}, Id_E)(X)(\mathcal{E}_L),$$

for any $X \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Particularly, we obtain:

$$\omega_L(S, \tilde{\partial}_b) = -\Gamma(\tilde{\rho}, Id_E)(\tilde{\partial}_b)(\mathcal{E}_L).$$

If we expand this equality, we obtain

$$\begin{aligned} &\left(g_f^d \circ h \circ \pi \right) y^f \left[(\rho_d^i \circ h \circ \pi) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial x^i} - (\rho_b^i \circ h \circ \pi) \frac{\partial((\tilde{g}_d^e \circ h \circ \pi) L_e)}{\partial x^i} \right. \\ &\quad \left. - (L_{ab}^c \circ h \circ \pi) (\tilde{g}_c^e \circ h \circ \pi) L_e \right] - 2 \left(G^a - \frac{1}{4} F^a \right) (\tilde{g}_a^e \circ h \circ \pi) \cdot L_{eb} \\ &= \rho_b^i \circ h \circ \pi \cdot L_i - (\rho_b^i \circ h \circ \pi) \frac{\partial((g_e^a \circ h \circ \pi) y^e L_a)}{\partial x^i}. \end{aligned}$$

After some calculations, we obtain the conclusion of the theorem. *q.e.d.*

Remark 9.1 If $F_e = 0$ and $\eta = Id_M$, then

$$E_b(L, Id_E, Id_M) = (\rho_b^i \circ \pi) L_i - y^d (\rho_b^i \circ \pi) L_{id} + y^d (L_{db}^c \circ \pi) L_c$$

and $S_L(Id_E, Id_M) \stackrel{put}{=} S_L$ is the canonical ρ -semispray associated to regular Lagrangian L which is similar with the semispray presented in [9] by M. de Leon, J. Marrero and E. Martinez.

In addition, if $F_e \neq 0$ and $(\rho, \eta) = (Id_{TM}, Id_M)$, then $S_L(Id_E, Id_M) \stackrel{put}{=} S_L$ will be called *the canonical semispray* which is similar with the semispray presented by I. Bucataru and R. Miron in [5].

In particular, if $F_e = 0$ and $(\rho, \eta) = (Id_{TM}, Id_M)$, then $S_L(Id_M, Id_E) \stackrel{put}{=} S_L$ will be called *the canonical semispray* which is similar with the canonical semispray presented by R. Miron and M. Anastasiei in [13]. (see also [14])

Theorem 9.2 *If $S_L(g, h)$ is the canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) , then the real local functions*

$$(9.9) \quad \begin{aligned} (\rho, \eta) \Gamma_c^a &= -\frac{1}{2} (\tilde{g}_c^d \circ h \circ \pi) \frac{\partial (E_b(L, g, h) \tilde{L}^{be} (g_e^a \circ h \circ \pi))}{\partial y^d} \\ &\quad -\frac{1}{2} (g_e^d \circ h \circ \pi) y^e \left(L_{dc}^f \circ h \circ \pi \right) (\tilde{g}_f^a \circ h \circ \pi) \\ &\quad +\frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad -\frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial (\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} \end{aligned}$$

are the components of a (ρ, η) -connection $(\rho, \eta) \Gamma$ for the vector bundle (E, π, M) which will be called the (ρ, η) -connection associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .

In particular, if $\eta = h = Id_M$ and $g = Id_E$, then we obtain

$$(9.9') \quad \rho \Gamma_c^a = -\frac{1}{2} \frac{\partial (E_b(L, Id_E, Id_M) \tilde{L}^{ba})}{\partial y^c} - \frac{1}{2} y^b L_{bc}^a \circ \pi.$$

Theorem 9.3 *The parallel (g, h) -lifts with respect to (ρ, η) -connection $(\rho, \eta) \Gamma$ are the integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .*

Definition 9.4 The equations

$$(9.10) \quad \frac{dy^a(t)}{dt} - \left(E_b(L, g, h) \tilde{L}^{be} (g_e^a \circ h \circ \pi) \right) \circ u(c, \dot{c})(x(t)) = 0,$$

where $x(t) = \eta \circ h \circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .*

The equations

$$(9.10') \quad \frac{dy^a(t)}{dt} - \left(E_b(L, Id_E, Id_M) \tilde{L}^{ba} \right) \circ u(c, \dot{c})(x(t)) = 0,$$

where $x(t) = c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$.*

Remark 9.1 The integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^V -morphism (g, h) are the (g, h) -lifts solutions for the equations of Euler-Lagrange type (9.10).

Using our theory, we obtain the following

Theorem 9.4 *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g, h) -lifts are solutions for the equations of Euler-Lagrange type (9.10).*

Therefore, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid $((TM, \tau_M, M), [\cdot, \cdot]_{TM}, (Id_{TM}, Id_M))$, to an arbitrary (generalized) Lie algebroid $((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta))$.

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LAGRANGIAN MECHANICS ON GENERALIZED LIE ALGEBROIDS

by
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Abstract

A solution for the Weinstein's Problem in the general framework of generalized Lie algebroids is the target of this paper. We present the *mechanical systems* called by use, *mechanical (ρ, η) -systems*, *Lagrange mechanical (ρ, η) -systems* or *Finsler mechanical (ρ, η) -systems* and we develop their geometries. We obtain the canonical (ρ, η) -semi(spray) associated to a mechanical (ρ, η) -system. The Lagrange mechanical (ρ, η) -systems are the spaces necessary to develop a Lagrangian formalism. We obtain the (ρ, η) -semispray associated to a regular Lagrangian L and external force F_e and we derive the equations of Euler-Lagrange type. A new point of view over classical and modern results about Lagrangian Mechanics are presented.

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Keywords: fiber bundle, vector bundle, (generalized) Lie algebroid, (linear) connection, curve, lift, natural base, adapted base, projector, almost product structure, almost tangent structure, complex structure, spray, semispray, mechanical system, Lagrangian formalism.

Contents

1	Introduction	2
2	Preliminaries	3
3	Natural and adapted basis	6
4	The lift of a differentiable curve	11
5	Remarkable Mod-endomorphisms	13
5.1	Projectors	13
5.2	The almost product structure	15
5.3	The almost tangent structure	16
6	Tensor d-fields. Distinguished linear (ρ, η)-connections	16
7	Mechanical systems	20
8	(ρ, η)-semisprays and (ρ, η)-sprays for mechanical (ρ, η)-systems	21
9	A Lagrangian formalism for Lagrange mechanical (ρ, η)-systems	28
	References	31

1 Introduction

The notion of Lagrange space was introduced and studied by J. Kern [7] and R. Miron [12]. The geometry of Lagrange spaces was extensively examined by geometers and physicists from Canada, Germany, Hungary, Italy, Japan, Romania, Russia and USA. Many international conferences were devoted to debate this subject, proceedings and monographs were published [4, 5, 13, 14, 15, 18]. In the classical sense, a regular Lagrangian on TM is a smooth function $TM \xrightarrow{L} \mathbb{R}$ such that the Hessian matrix with entries

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 H(x, y)}{\partial y^i \partial y^j}$$

is everywhere nondegenerate on TM (or on a domain of TM) and a Lagrange space is a pair $L^n = (M, L)$, where L is a regular Lagrangian (see [15]).

We know that a geodesic of the Lagrange space L^n is an extremal curve of the action integral

$$I(c) = \int_0^1 L\left(x(t), \frac{dx(t)}{dt}\right) dt.$$

This is, in fact, a solution of the Euler-Lagrange system of equations:

$$\dot{x}^i = \frac{dx^i(t)}{dt}, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0,$$

where $(x^i(t), i \in \overline{1, m})$ are the local coordinates of the point $c(t)$.

This system is equivalent to

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0.$$

Here $G^i, i \in \overline{1, m}$ are the components of a semispray that generates a notable nonlinear connection, whose coefficients are given by

$$N_k^i = \frac{\partial G^i}{\partial y^k}.$$

The case when L is square of a function on TM , positively, 1-homogeneous with respect to the velocity y^i , provides an important class of Lagrange spaces called Finsler spaces.

We know the **Weinstein's Problem**:

Develop a Lagrangian formalism directly on the given Lie algebroid similar to Klein's formalism for ordinary Lagrangian Mechanics (see [8]).

This problem was formulated by A. Weinstein in [23], where the author gave the theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the dual of a Lie algebroid and the Legendre transformation defined by a regular Lagrangian. In [10], P. Liberman showed that such a formalism is not possible if one consider the tangent bundle of a Lie algebroid as space for developing the theory. Using the prolongation of a Lie algebroid over a smooth map, E. Martinez solved the **Weinstein's Problem** in [11] (see also [6, 9]).

A Lagrangian description of Mechanics on Lie algebroids was extensively studied by many authors. (see [16, 17, 19, 20, 21, 22])

The generalized Lie algebroid is a new notion necessary to obtain a new class of (linear) connections in Ehresmann sense.(see [1]) The notions of *IDS* and *EDS* for Lie algebroids presented in [3] was natural extended to generalized Lie algebroids in [2]. In particular there are obtained a new point of view over exterior differential calculus for Lie algebroids.

In this paper we propose to solve the **Weinstein's Problem** in the general framework of generalized Lie algebroids.

In the Sections 3,4,5 and 6 we set up the basic notions and terminology. In Section 7 we introduce the *mechanical systems* called by use, *mechanical* (ρ, η) -systems, *Lagrange mechanical* (ρ, η) -systems or *Finsler mechanical* (ρ, η) -systems. In Section 8 we present the *canonical* (ρ, η) -semispray associated to *mechanical* (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) . If $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to a connection Γ presented by I. Bucataru and R. Miron in [5]. Also, we obtain the *canonical* (ρ, η) -spray associated to *mechanical system* $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) .

The Section 9 is dedicated to study the geometry of Lagrange mechanical (ρ, η) -systems. These mechanical systems are the spaces necessary to solve the **Weinstein's Problem** in the general framework of generalized Lie algebroids. We determine and we study the (ρ, η) -semispray associated to a regular Lagrangian L and external force F_e which are applied on the total space of a generalized Lie algebroid and we derive the equations of Euler-Lagrange type. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the **Weinstein's Problem**, different by the Martinez's solution.

Finally, we obtain that the integral curves of the canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) are the (g, h) -lifts solutions for the equations of Euler-Lagrange type (9.10).

Using our theory, we obtain the following

Theorem *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g, h) -lifts are solutions for the equations of Euler-Lagrange type (9.10).*

As any Lie algebroid can be regarded as a particularly generalized Lie algebroid, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid $((TM, \tau_M, M), [,]_{TM}, (Id_{TM}, Id_M))$, to an arbitrary (generalized) Lie algebroid $((E, \pi, M), [,]_{E, h}, (\rho, \eta))$.

2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and \mathbf{B}^\vee be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B}^\vee|$ so that M is paracompact and if $A \subseteq M$ is closed, then for any section u over A it exists $\tilde{u} \in \Gamma(E, \pi, M)$ so that $\tilde{u}|_A = u$. In the following, we consider only vector bundles with paracompact base.

Additionally, if $(E, \pi, M) \in |\mathbf{B}^\vee|$, $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$ -module. If $(\varphi, \varphi_0) \in$

$\mathbf{B}^\vee((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in \text{Iso}_{\mathbf{Man}}(M, M')$, then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad \cdot \quad} & \Gamma(E', \pi', M') \\ (f, u') & \mapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that $(\Gamma(E', \pi', M'), +, \cdot)$ is a $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \mapsto & \Gamma(\varphi, \varphi_0)u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0)u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any $y \in M'$.

Let $M, N \in |\mathbf{Man}|$, $h \in \text{Iso}_{\mathbf{Man}}(M, N)$ and $\eta \in \text{Iso}_{\mathbf{Man}}(N, M)$.

We know (see [2, 3]) that if $(F, \nu, N) \in |\mathbf{B}^\vee|$ so that there exists

$$(\rho, \eta) \in \mathbf{B}^\vee((F, \nu, N), (TM, \tau_M, M))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \mapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

*GLA*₁. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$.

*GLA*₂. the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$ is a Lie $\mathcal{F}(N)$ -algebra,

*GLA*₃. the **Mod**-morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is a **LieAlg**-morphism of

$$\left(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h}\right)$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$$

target, then the triple $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is called generalized Lie algebroid.

In particular, if $h = Id_M = \eta$, then we obtain the definition of the Lie algebroid.

We can discuss about *the category GLA of generalized Lie algebroids* [2].

Examples of objects of this category are presented in the paper [1].

Let $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ be an object of the category **GLA**.

- Locally, for any $\alpha, \beta \in \overline{1, p}$, we set $[t_\alpha, t_\beta]_{F,h} = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha, \beta, \gamma \in \overline{1, p}$.

The real local functions $L_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma \in \overline{1, p}$ will be called the *structure functions of the generalized Lie algebroid* $\left((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta)\right)$.

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\ (\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}}) \end{array}$$

where $i, \tilde{i} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$\begin{aligned} (\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'} (\chi^{\tilde{i}}), z^{\alpha'} (\chi^{\tilde{i}}, z^\alpha)), \\ (x^i, y^i) &\longrightarrow (x^{\tilde{i}'} (x^i), y^{\tilde{i}'} (x^i, y^i)) \end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'} (\chi^{\tilde{i}}), z^{\tilde{i}'} (\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$ is arbitrary, then

$$\begin{aligned} (2.1) \quad & \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha) f(h \circ \eta(\varkappa)) = \\ & = \left(\theta_\alpha^{\tilde{i}} z^\alpha \frac{\partial f}{\partial x^{\tilde{i}}} \right) (h \circ \eta(\varkappa)) = \left((\rho_\alpha^i \circ h)(z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)), \end{aligned}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_α^i respectively $\theta_\alpha^{\tilde{i}}$ change to $\rho_{\alpha'}^{\tilde{i}'}$ respectively $\theta_{\alpha'}^{\tilde{i}'}$ according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \rho_\alpha^i \frac{\partial x^{\tilde{i}'}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}'} = \Lambda_\alpha^{\alpha'} \theta_\alpha^{\tilde{i}} \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}},$$

where

$$\|\Lambda_\alpha^{\alpha'}\| = \left\| \Lambda_\alpha^{\alpha'} \right\|^{-1}.$$

Remark 2.1 The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_\alpha^{\tilde{i}} \frac{\partial f}{\partial \chi^{\tilde{i}}} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left(L_{\alpha\beta}^\gamma \circ h \right) \left(\rho_\gamma^k \circ h \right) = \left(\rho_\alpha^i \circ h \right) \frac{\partial \left(\rho_\beta^k \circ h \right)}{\partial x^i} - \left(\rho_\beta^j \circ h \right) \frac{\partial \left(\rho_\alpha^k \circ h \right)}{\partial x^j}.$$

We have the \mathbf{B}^\vee -morphism

$$(2.6) \quad \begin{array}{ccc} \pi^*(h^*F) & \hookrightarrow & F \\ \pi^*(h^*\nu) \downarrow & & \downarrow \nu \\ M & \xrightarrow{h \circ \pi} & N \end{array}$$

Let $\left(\begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right)$ be the \mathbf{B}^\vee -morphism of $(\pi^*(h^*F), \pi^*(h^*\nu), E)$ source and (TE, τ_E, E) target, where

$$(2.7) \quad \begin{array}{ccc} \pi^*(h^*F) & \xrightarrow{\begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}} & TE \\ Z^\alpha T_\alpha(u_x) & \longmapsto & \left(Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi \right) \frac{\partial}{\partial x^i}(u_x) \end{array}$$

Using the operation

$$\Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)^2 \xrightarrow{[\cdot]_{\pi^*(h^*F)}} \Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)$$

defined by

$$(2.8) \quad \begin{aligned} [T_\alpha, T_\beta]_{\pi^*(h^*F)} &= \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) T_\gamma, \\ [T_\alpha, fT_\beta]_{\pi^*(h^*F)} &= f \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) T_\gamma + \left(\rho_\alpha^i \circ h \circ \pi \right) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{\pi^*(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^*(h^*F)}, \end{aligned}$$

for any $f \in \mathcal{F}(E)$, it results that

$$\left((\pi^*(h^*F), \pi^*(h^*\nu), E), [\cdot]_{\pi^*(h^*F)}, \left(\begin{smallmatrix} \pi^*(h^*F) \\ \rho \end{smallmatrix}, Id_E \right) \right)$$

is a Lie algebroid.

3 Natural and adapted basis

We consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} E & & \left((F, [\cdot]_{F,h}, (\rho, \eta)) \right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where (E, π, M) is a vector bundle and $\left((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta) \right)$ is a generalized Lie algebroid.

We take (x^i, y^a) as canonical local coordinates on (E, π, M) , where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Let

$$(x^i, y^a) \longrightarrow (x^{\acute{i}}(x^i), y^{\acute{a}}(x^i, y^a))$$

be a change of coordinates on (E, π, M) . Then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$(3.2) \quad y^{\acute{a}} = M_a^{\acute{a}} y^a.$$

Let

$$(3.3) \quad \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right) \stackrel{put}{=} \left(\partial_i, \dot{\partial}_a \right)$$

be the natural base of the Lie algebroid $((TE, \tau_E, E), [\cdot, \cdot]_{TE}, (Id_{TE}, Id_E))$.

For any sections

$$Z^\alpha T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*F), E)$$

and

$$Y^a \dot{\partial}_a \in \Gamma(VTE, \tau_E, E)$$

we obtain the section

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a &=: Z^\alpha (T_\alpha \oplus (\rho_\alpha^i \circ h \circ \pi) \partial_i) + Y^a (0_{\pi^*(h^*F)} \oplus \dot{\partial}_a) \\ &= Z^\alpha T_\alpha \oplus \left(Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a \right) \in \Gamma \left(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E \right). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a &= 0 \\ \Updownarrow \\ Z^\alpha T_\alpha &= 0 \wedge Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a = 0, \end{aligned}$$

it implies $Z^\alpha = 0$, $\alpha \in \overline{1, p}$ and $Y^a = 0$, $a \in \overline{1, r}$.

Therefore, the sections $\tilde{\partial}_1, \dots, \tilde{\partial}_p, \dot{\tilde{\partial}}_1, \dots, \dot{\tilde{\partial}}_r$ are linearly independent.

We consider the vector subbundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ of the vector bundle $(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E)$, for which the $\mathcal{F}(E)$ -module of sections is the $\mathcal{F}(E)$ -submodule of $(\Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E), +, \cdot)$, generated by the set of sections $(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a)$ which is called the *natural* (ρ, η) -base.

The matrix of coordinate transformation on $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_\alpha^{\acute{\alpha}} \circ h \circ \pi & 0 \\ (\rho_a^i \circ h \circ \pi) \frac{\partial M_b^{\acute{a}} \circ \pi}{\partial x_i} y^b & M_a^{\acute{a}} \circ \pi \end{array} \right\|.$$

We have the following

Theorem 3.1 Let $(\tilde{\rho}, Id_E)$ be the $\mathbf{B}^{\mathbf{v}}$ -morphism of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (TE, τ_E, E) target, where

$$(3.5) \quad \begin{aligned} & (\rho, \eta) TE \xrightarrow{\tilde{\rho}} TE \\ & \left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \longmapsto \left(Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a \right) (u_x) \end{aligned}$$

Using the operation

$$\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)^2 \xrightarrow{[\cdot]_{(\rho, \eta) TE}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

defined by

$$(3.6) \quad \begin{aligned} & \left[\left(Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right), \left(Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right) \right]_{(\rho, \eta) TE} \\ & = \left[Z_1^\alpha T_\alpha, Z_2^\beta T_\beta \right]_{\pi^*(h^* F)} \oplus \left[(\rho_\alpha^i \circ h \circ \pi) Z_1^\alpha \partial_i + Y_1^a \dot{\partial}_a, \right. \\ & \quad \left. (\rho_\beta^j \circ h \circ \pi) Z_2^\beta \partial_j + Y_2^b \dot{\partial}_b \right]_{TE}, \end{aligned}$$

for any $\left(Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right)$ and $\left(Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right)$, we obtain that the couple

$$([\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E))$$

is a Lie algebroid structure for the vector bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

The Lie algebroid

$$\left(((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [\cdot]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right),$$

is called the *Lie algebroid generalized tangent bundle*.

Remark 3.1 The following equalities hold good:

$$(3.7) \quad \begin{aligned} \left[\tilde{\partial}_\alpha, \tilde{\partial}_\beta \right]_{(\rho, \eta) TE} &= L_{\alpha\beta}^\gamma \circ h \circ \pi \cdot \tilde{\partial}_\gamma \\ \left[\tilde{\partial}_\alpha, \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= 0_{(\rho, \eta) TE} \\ \left[\tilde{\partial}_a, \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= 0_{(\rho, \eta) TE} \end{aligned}$$

We consider the $\mathbf{B}^{\mathbf{v}}$ -morphism $((\rho, \eta) \pi!, Id_E)$ given by the commutative diagram

$$(3.8) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^*(h^* F) \\ (\rho, \eta) \tau_E \downarrow & & \downarrow pr_1 \\ E & \xrightarrow{id_E} & E \end{array}$$

This is defined as:

$$(3.9) \quad (\rho, \eta) \pi! \left(\left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) \right) = (Z^\alpha T_\alpha) (u_x),$$

for any $\left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a\right) \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Using the \mathbf{B}^V -morphisms (2.6) and (3.7) we obtain the *tangent* (ρ, η) -application $((\rho, \eta) T\pi, h \circ \pi)$ of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (F, ν, N) target.

Definition 3.1 The kernel of the tangent (ρ, η) -application is written

$$(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and it is called *the vertical subbundle*.

We remark that the set $\left\{\dot{\tilde{\partial}}_a, a \in \overline{1, r}\right\}$ is a base of the $\mathcal{F}(E)$ -module

$$(\Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot).$$

Proposition 3.1 *The short sequence of vector bundles*

$$(3.10) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta) TE & \xrightarrow{i} & (\rho, \eta) TE & \xrightarrow{(\rho, \eta) \pi!} & \pi^*(h^* F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E \end{array}$$

is exact.

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle (E, π, M) , i. e. a **Man**-morphism $(\rho, \eta) \Gamma$ of $(\rho, \eta) TE$ source and $V(\rho, \eta) TE$ target defined by

$$(3.11) \quad (\rho, \eta) \Gamma \left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) (u_x) = (Y^a + (\rho, \eta) \Gamma_\alpha^a Z^\alpha) \dot{\tilde{\partial}}_a (u_x),$$

so that the \mathbf{B}^V -morphism $((\rho, \eta) \Gamma, Id_E)$ is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.12) \quad (\rho, \eta) \Gamma_\gamma^{a'} = M_a^{a'} \circ \pi \left[\rho_\gamma^i \circ (h \circ \pi) \frac{\partial M_b^{a'} \circ \pi}{\partial x^i} y^b + (\rho, \eta) \Gamma_\gamma^a \right] \Lambda_\gamma^{\gamma'} \circ (h \circ \pi).$$

The kernel of the \mathbf{B}^V -morphism $((\rho, \eta) \Gamma, Id_E)$ is written $(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ and is called the *horizontal vector subbundle*.

We remark that the horizontal and the vertical vector subbundles are interior differential systems of the Lie algebroid generalized tangent bundle. (see [4])

We put the problem of finding a base for the $\mathcal{F}(E)$ -module

$$(\Gamma(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$$

of the type

$$\tilde{\delta}_\alpha = Z_\alpha^\beta \tilde{\partial}_\beta + Y_\alpha^a \dot{\tilde{\partial}}_a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.13) \quad \begin{aligned} \Gamma((\rho, \eta) \pi!, Id_E) \left(\tilde{\delta}_\alpha \right) &= T_\alpha, \\ \Gamma((\rho, \eta) \Gamma, Id_E) \left(\tilde{\delta}_\alpha \right) &= 0. \end{aligned}$$

Then we obtain the sections

$$(3.14) \quad \frac{\delta}{\delta \bar{z}^\alpha} = \tilde{\partial}_\alpha - (\rho, \eta) \Gamma_\alpha^a \dot{\tilde{\partial}}_a = T_\alpha \oplus \left((\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma_\alpha^a \dot{\partial}_a \right).$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a \right)$ will be called the *adapted* (ρ, η) -base.

Remark 3.2 The following equality holds good

$$(3.15) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right) = (\rho_\alpha^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma_\alpha^a \dot{\partial}_a.$$

Moreover, if $(\rho, \eta) \Gamma$ is the (ρ, η) -connection associated to a connection Γ (see [1]), then we obtain

$$(3.16) \quad \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right) = (\rho_\alpha^i \circ h \circ \pi) \delta_i,$$

where $\left(\delta_i, \dot{\partial}_a \right)$ is the adapted base for the $\mathcal{F}(E)$ -module $(\Gamma(TE, \tau_E, E), +, \cdot)$.

Theorem 3.2 *The following equality holds good*

$$(3.17) \quad \left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta \right]_{(\rho, \eta)TE} = L_{\alpha\beta}^\gamma \circ (h \circ \pi) \tilde{\delta}_\gamma + (\rho, \eta, h) \mathbb{R}_{\alpha\beta}^a \dot{\tilde{\partial}}_a,$$

where

$$(3.18) \quad \begin{aligned} (\rho, \eta, h) \mathbb{R}_{\alpha\beta}^a &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\beta \right) ((\rho, \eta) \Gamma_\alpha^a) \\ &\quad - \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right) ((\rho, \eta) \Gamma_\beta^a) + (L_{\alpha\beta}^\gamma \circ h \circ \pi) (\rho, \eta) \Gamma_\gamma^a, \end{aligned}$$

Moreover, we have:

$$(3.19) \quad \left[\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} = \Gamma(\tilde{\rho}, Id_E) \left(\dot{\tilde{\partial}}_b \right) ((\rho, \eta) \Gamma_\alpha^a) \dot{\tilde{\partial}}_a,$$

and

$$(3.20) \quad \Gamma(\tilde{\rho}, Id_E) \left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta \right]_{(\rho, \eta)TE} = \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\alpha \right), \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\beta \right) \right]_{TE}.$$

Let $(d\tilde{z}^\alpha, d\tilde{y}^b)$ be the natural dual (ρ, η) -base of natural (ρ, η) -base $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a \right)$.

This is determined by the equations

$$\begin{cases} \langle d\tilde{z}^\alpha, \tilde{\delta}_\beta \rangle = \delta_\beta^\alpha, & \langle d\tilde{z}^\alpha, \dot{\tilde{\partial}}_a \rangle = 0, \\ \langle d\tilde{y}^a, \tilde{\delta}_\beta \rangle = 0, & \langle d\tilde{y}^a, \dot{\tilde{\partial}}_b \rangle = \delta_b^a. \end{cases}$$

We consider the problem of finding a base for the $\mathcal{F}(E)$ -module

$$(\Gamma((V(\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E), +, \cdot)$$

of the type

$$\delta \tilde{y}^a = \theta_\alpha^a d\tilde{z}^\alpha + \omega_b^a d\tilde{y}^b, \quad a \in \overline{1, n}$$

which satisfies the following conditions:

$$(3.21) \quad \left\langle \delta \tilde{y}^a, \dot{\tilde{\partial}}_a \right\rangle = 1 \wedge \left\langle \delta \tilde{y}^a, \tilde{\delta}_\alpha \right\rangle = 0.$$

We obtain the sections

$$(3.22) \quad \delta \tilde{y}^a = (\rho, \eta) \Gamma_\alpha^a d\tilde{z}^\alpha + d\tilde{y}^a, \quad a \in \overline{1, n}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base $(d\tilde{z}^\alpha, \delta \tilde{y}^a)$ will be called the *adapted dual* (ρ, η) -base.

4 The lift of a differentiable curve

We consider the following diagram:

$$(4.1) \quad \begin{array}{ccc} E & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) & \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^v|$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|$.

We admit that $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) and $I \xrightarrow{c} M$ is a differentiable curve. We know that

$$(E|_{\text{Im}(\eta \circ h \circ c)}, \pi|_{\text{Im}(\eta \circ h \circ c)}, \text{Im}(\eta \circ h \circ c))$$

is a vector subbundle of the vector bundle (E, π, M) .

Definition 4.1 If

$$(4.2) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & E|_{\text{Im}(\eta \circ h \circ c)} \\ t & \longmapsto & y^a(t) s_a(\eta \circ h \circ c(t)) \end{array}$$

is a differentiable curve such that there exists $g \in \mathbf{Man}(E, F)$ such that the following conditions are satisfied:

1. $(g, h) \in \mathbf{B}^v((E, \pi, M), (F, \nu, N))$ and
2. $\rho \circ g \circ \dot{c}(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt} \frac{\partial}{\partial x^i}((\eta \circ h \circ c)(t))$, for any $t \in I$,

then we will say that \dot{c} is the (g, h) -lift of the differentiable curve c .

Remark 4.1 The condition 2 is equivalent with the following affirmation:

$$(4.3) \quad \rho_\alpha^i(\eta \circ h \circ c(t)) \cdot g_\alpha^i(h \circ c(t)) \cdot y^a(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt}, \quad i \in \overline{1, m}.$$

Definition 4.2 If $I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)}$ is a differentiable (g, h) -lift of the differentiable curve c , then the section

$$(4.4) \quad \begin{array}{ccc} \text{Im}(\eta \circ h \circ c) & \xrightarrow{u(c, \dot{c})} & E|_{\text{Im}(\eta \circ h \circ c)} \\ \eta \circ h \circ c(t) & \longmapsto & \dot{c}(t) \end{array}$$

will be called the *canonical section associated to the couple (c, \dot{c})* .

Definition 4.3 If $(g, h) \in \mathbf{B}^{\mathbf{v}}((E, \pi, M), (F, \nu, N))$ has the components

$$g_a^\alpha; a \in \overline{1, r}, \alpha \in \overline{1, p}$$

such that for any vector local $(n + p)$ -chart (V, t_V) of (F, ν, N) there exists the real functions

$$V \xrightarrow{\tilde{g}_\alpha^a} \mathbb{R}; a \in \overline{1, r}, \alpha \in \overline{1, p}$$

such that

$$(4.5) \quad \tilde{g}_\alpha^b(\varkappa) \cdot g_a^\alpha(\varkappa) = \delta_a^b,$$

for any $\varkappa \in V$, then we will say that the $\mathbf{B}^{\mathbf{v}}$ -morphism (g, h) is *locally invertible*.

Remark 4.2 In particular, if $(Id_{TM}, Id_M, Id_M) = (\rho, \eta, h)$ and the $\mathbf{B}^{\mathbf{v}}$ morphism (g, Id_M) is locally invertible, then we have the differentiable (g, Id_M) -lift

$$(4.6) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \tilde{g}_j^i(c(t)) \frac{dc^j(t)}{dt} \frac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Moreover, if $g = Id_{TM}$, then we obtain the usual lift of tangent vectors

$$(4.7) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \frac{dc^i(t)}{dt} \frac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Definition 4.4 If $I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)}$ is a differentiable (g, h) -lift of differentiable curve c , such that its components functions $(y^a, a \in \overline{1, n})$ are solutions for the differentiable system of equations:

$$(4.8) \quad \frac{du^a}{dt} + (\rho, \eta) \Gamma_\alpha^a \circ u(c, \dot{c}) \circ (\eta \circ h \circ c) \cdot g_b^\alpha \circ h \circ c \cdot u^b = 0,$$

then we will say that the (g, h) -lift \dot{c} is *parallel with respect to the (ρ, η) -connection $(\rho, \eta) \Gamma$* .

Remark 4.3 In particular, if $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$ and the $\mathbf{B}^{\mathbf{v}}$ morphism (g, Id_M) is locally invertible, then the differentiable (g, Id_{TM}) -lift

$$(4.9) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \left(\tilde{g}_j^i \circ c \cdot \frac{dc^j}{dt} \right) \frac{\partial}{\partial x^i}(c(t)), \end{array}$$

is parallel with respect to the connection Γ if the component functions

$$\left(\tilde{g}_j^i \circ c \cdot \frac{dc^j}{dt}, i \in \overline{1, n} \right)$$

are solutions for the differentiable system of equations

$$(4.10) \quad \frac{du^i}{dt} + \Gamma_k^i \circ u(c, \dot{c}) \circ c \cdot g_h^k \circ c \cdot u^h = 0,$$

namely

$$(4.10)' \quad \begin{aligned} & \frac{d}{dt} \left(\tilde{g}_j^i(c(t)) \cdot \frac{dc^j(t)}{dt} \right) \\ & + \Gamma_k^i \left(\left(\tilde{g}_j^i(c(t)) \cdot \frac{dc^j(t)}{dt} \right) \cdot \frac{\partial}{\partial x^i}(c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0. \end{aligned}$$

Moreover, if $g = Id_{TM}$, then the usual lift of tangent vectors $(4.6)'$ is parallel with respect to the connection Γ if the component functions $\left(\frac{dc^j}{dt}, j \in \overline{1, n} \right)$ are solutions for the differentiable system of equations

$$(4.11) \quad \frac{du^i}{dt} + \Gamma_k^i \circ u(c, \dot{c}) \circ c \cdot u^k = 0,$$

namely

$$(4.11)' \quad \frac{d}{dt} \left(\frac{dc^j(t)}{dt} \right) + \Gamma_k^i \left(\frac{dc^j(t)}{dt} \cdot \frac{\partial}{\partial x^i}(c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0.$$

5 Remarkable Mod-endomorphisms

In the following we consider the diagram:

$$\begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^v|$ and $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid.

Definition 5.1 For any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ we define the application of Nijenhuis type

$$\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)^2 \xrightarrow{N_e} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

defined by

$$N_e(X, Y) = [eX, eY]_{\rho TE} + e^2[X, Y]_{\rho TE} - e[eX, Y]_{\rho TE} - e[X, eY]_{\rho TE},$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

5.1 Projectors

Definition 5.1.1 Any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the property

$$(5.1.1) \quad e^2 = e$$

will be called *projector*.

Example 5.1.1 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) & \xrightarrow{\mathcal{V}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a & \longmapsto Y^a \dot{\tilde{\partial}}_a \end{aligned}$$

is a projector which will be called *the vertical projector*.

Remark 5.1.1 We have $\mathcal{V}(\tilde{\delta}_\alpha) = 0$ and $\mathcal{V}(\dot{\tilde{\partial}}_a) = \dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{V}(\dot{\tilde{\partial}}_\alpha) = (\rho, \eta) \Gamma_\alpha^a \dot{\tilde{\partial}}_a.$$

In addition, we obtain the equality

$$(5.1.2) \quad \Gamma((\rho, \eta) \Gamma, Id_E) \left(Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \right) = \mathcal{V} \left(Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \right),$$

for any $Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Theorem 5.1.1 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{V} of $\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ with the properties:

$$(5.1.3) \quad \begin{aligned} \mathcal{V}(\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)) & \subset \Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\ \mathcal{V}(X) = X & \iff X \in \Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E) \end{aligned}$$

Example 5.1.2 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) & \xrightarrow{\mathcal{H}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a & \longmapsto Z^\alpha \tilde{\delta}_\alpha \end{aligned}$$

is a projector which will be called *the horizontal projector*.

Remark 5.1.2 We have $\mathcal{H}(\tilde{\delta}_\alpha) = \tilde{\delta}_\alpha$ and $\mathcal{H}(\dot{\tilde{\partial}}_a) = 0$. Therefore, we obtain $\mathcal{H}(\dot{\tilde{\partial}}_\alpha) = \tilde{\delta}_\alpha$.

Theorem 5.1.2 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{H} of $\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ with the properties:

$$(5.1.4) \quad \begin{aligned} \mathcal{H}(\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)) & \subset \Gamma(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\ \mathcal{H}(X) = X & \iff X \in \Gamma(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E). \end{aligned}$$

Corollary 5.1.1 A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{H} of $\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ with the properties:

$$(5.1.5) \quad \begin{aligned} \mathcal{H}^2 &= \mathcal{H} \\ Ker(\mathcal{H}) &= (\Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot). \end{aligned}$$

Remark 5.1.3 For any $X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ we obtain the unique decomposition

$$X = \mathcal{H}X + \mathcal{V}X.$$

Proposition 5.1.1 *After some calculations we obtain*

$$(5.1.6) \quad N_{\mathcal{V}}(X, Y) = \mathcal{V}[\mathcal{H}X, \mathcal{H}Y]_{(\rho, \eta)TE} = N_{\mathcal{H}}(X, Y),$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Corollary 5.1.2 *The horizontal interior differential system*

$$(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

is involutive if and only if $N_{\mathcal{V}} = 0$ or $N_{\mathcal{H}} = 0$.

5.2 The almost product structure

Definition 5.2.1 Any **Mod**-endomorphism e of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the property

$$(5.2.1) \quad e^2 = Id$$

will be called the *almost product structure*.

Example 5.2.1 The **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{P}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a &\longmapsto Z^\alpha \tilde{\delta}_\alpha - Y^a \dot{\tilde{\partial}}_a \end{aligned}$$

is an almost product structure.

Remark 5.2.1 The previous almost product structure has the properties:

$$(5.2.2) \quad \begin{aligned} \mathcal{P} &= (2\mathcal{H} - Id); \\ \mathcal{P} &= (Id - 2\mathcal{V}); \\ \mathcal{P} &= (\mathcal{H} - \mathcal{V}). \end{aligned}$$

Remark 5.2.2 We obtain that $\mathcal{P}(\tilde{\delta}_\alpha) = \tilde{\delta}_\alpha$ and $\mathcal{P}(\dot{\tilde{\partial}}_a) = -\dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{P}(\dot{\tilde{\partial}}_a) = \tilde{\delta}_\alpha - \rho \Gamma_\alpha^a \dot{\tilde{\partial}}_a.$$

Theorem 5.2.1 *A (ρ, η) -connection for the vector bundle (E, π, M) is characterized by the existence of a **Mod**-endomorphism \mathcal{P} of $\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the following property:*

$$(5.2.3) \quad \mathcal{P}(X) = -X \iff X \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E).$$

Proposition 5.2.1 *After some calculations, we obtain*

$$N_{\mathcal{P}}(X, Y) = 4\mathcal{V}[\mathcal{H}X, \mathcal{H}Y],$$

for any $X, Y \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Corollary 5.2.1 *The horizontal interior differential system $(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is involutive if and only if $N_{\mathcal{P}} = 0$.*

5.3 The almost tangent structure

Definition 5.3.1 Any **Mod**-endomorphism e of $(\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ with the property

$$(5.3.1) \quad e^2 = 0$$

will be called the *almost tangent structure*.

Example 5.3.1 If $(E, \pi, M) = (F, \nu, N)$, $g \in \mathbf{Man}(E, E)$ such that (g, h) is a locally invertible \mathbf{B}^\vee -morphism, then the **Mod**-endomorphism

$$\begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{J}_{(g, h)}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ Z^a \tilde{\delta}_a + Y^b \dot{\tilde{\delta}}_b &\longmapsto (\tilde{g}_a^b \circ h \circ \pi) Z^a \dot{\tilde{\delta}}_b \end{aligned}$$

is an almost tangent structure which will be called the *almost tangent structure associated to the \mathbf{B}^\vee -morphism (g, h)* . (See: Definition 4.3)

Remark 5.3.1 We obtain that

$$\mathcal{J}_{(g, h)}(\tilde{\delta}_a) = \mathcal{J}_{(g, h)}(\dot{\tilde{\delta}}_a) = (\tilde{g}_a^b \circ h \circ \pi) \dot{\tilde{\delta}}_b \text{ and } \mathcal{J}_{(g, h)}(\dot{\tilde{\delta}}_b) = 0.$$

and we have the following properties:

$$(5.3.2) \quad \begin{aligned} \mathcal{J}_{(g, h)} \circ \mathcal{P} &= \mathcal{J}_{(g, h)}; \\ \mathcal{P} \circ \mathcal{J}_{(g, h)} &= -\mathcal{J}_{(g, h)}; \\ \mathcal{J}_{(g, h)} \circ \mathcal{H} &= \mathcal{J}_{(g, h)}; \\ \mathcal{H} \circ \mathcal{J}_{(g, h)} &= 0; \\ \mathcal{J}_{(g, h)} \circ \mathcal{V} &= 0; \\ \mathcal{V} \circ \mathcal{J}_{(g, h)} &= \mathcal{J}_{(g, h)}; \\ N_{\mathcal{J}_{(g, h)}} &= 0. \end{aligned}$$

6 Tensor d -fields. Distinguished linear (ρ, η) -connections

We consider the following diagram:

$$\begin{array}{ccc} E & & (F, [\cdot, \cdot]_{F, h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B}^\vee|$ and $((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta))$ is a generalized Lie algebroid.

Let

$$(\mathcal{T}_{q, s}^{p, r}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$$

be the $\mathcal{F}(E)$ -module of tensor fields by $(\frac{p, r}{q, s})$ -type from the generalized tangent bundle

$$(H(\rho, \eta)TE, (\rho, \eta)\tau_E, E) \oplus (V(\rho, \eta)TE, (\rho, \eta)\tau_E, E).$$

An arbitrarily tensor field T is written as

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}_{a_1} \otimes \dots \otimes \tilde{\partial}_{a_r} \otimes \delta\tilde{y}^{b_1} \otimes \dots \otimes \delta\tilde{y}^{b_s}.$$

Let

$$(\mathcal{T}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

If $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ and $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$, then the components of product tensor field $T_1 \otimes T_2$ are the products of local components of T_1 and T_2 .

Therefore, we obtain $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Let $\mathcal{DT}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ be the family of tensor fields

$$T \in \mathcal{T}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0}((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \text{ and } T_2 \in \mathcal{T}_{0,s}^{0,r}((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

such that $T = T_1 + T_2$.

The $\mathcal{F}(E)$ -module $(\mathcal{DT}((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot)$ will be called the *module of distinguished tensor fields* or the *module of tensor d -fields*.

Remark 5.1 The elements of

$$\Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

respectively

$$\Gamma(((\rho, \eta)TE)^*, ((\rho, \eta)\tau_E)^*, E)$$

are tensor d -fields.

Definition 6.1 Let (E, π, M) be a vector bundle endowed with a (ρ, η) -connection $(\rho, \eta)\Gamma$ and let

$$(6.1) \quad (X, T) \xrightarrow{(\rho, \eta)D} (\rho, \eta)D_X T$$

be a covariant (ρ, η) -derivative for the tensor algebra of the generalized tangent bundle

$$((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

which preserves the horizontal and vertical *IDS* by parallelism.

The real local functions

$$((\rho, \eta)H_{\beta\gamma}^\alpha, (\rho, \eta)H_{b\gamma}^a, (\rho, \eta)V_{\beta c}^\alpha, (\rho, \eta)V_{bc}^a)$$

defined by the following equalities:

$$(6.2) \quad \begin{aligned} (\rho, \eta)D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta &= (\rho, \eta)H_{\beta\gamma}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta)D_{\tilde{\delta}_\gamma} \tilde{\delta}_b &= (\rho, \eta)H_{b\gamma}^a \tilde{\delta}_a \\ (\rho, \eta)D_{\tilde{\partial}_c} \tilde{\delta}_\beta &= (\rho, \eta)V_{\beta c}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta)D_{\tilde{\partial}_c} \tilde{\delta}_b &= (\rho, \eta)V_{bc}^a \tilde{\delta}_a \end{aligned}$$

are the components of a linear (ρ, η) -connection

$$((\rho, \eta) H, (\rho, \eta) V)$$

for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ which will be called the *distinguished linear (ρ, η) -connection*.

If $h = Id_M$, then the distinguished linear (Id_{TM}, Id_M) -connection will be called the *distinguished linear connection*.

The components of a distinguished linear connection (H, V) will be denoted

$$(H_{jk}^i, H_{bk}^a, V_{jc}^i, V_{bc}^a).$$

Theorem 6.1 *If $((\rho, \eta)H, (\rho, \eta)V)$ is a distinguished linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, then its components satisfy the change relations:*

$$\begin{aligned} (\rho, \eta) H_{\beta\gamma}^{\alpha'} &= \Lambda_{\alpha}^{\alpha'} \circ h \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_{\gamma} \right) \left(\Lambda_{\beta}^{\alpha} \circ h \circ \pi \right) + \right. \\ &\quad \left. + (\rho, \eta) H_{\beta\gamma}^{\alpha} \cdot \Lambda_{\beta}^{\beta} \circ h \circ \pi \right] \cdot \Lambda_{\gamma}^{\gamma} \circ h \circ \pi, \\ (\rho, \eta) H_{b\gamma}^a &= M_a^a \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_{\gamma} \right) (M_b^a \circ \pi) + \right. \\ &\quad \left. + (\rho, \eta) H_{b\gamma}^a \cdot M_b^b \circ \pi \right] \cdot \Lambda_{\gamma}^{\gamma} \circ h \circ \pi, \\ (\rho, \eta) V_{\beta c}^{\alpha'} &= \Lambda_{\alpha}^{\alpha'} \circ h \circ \pi \cdot (\rho, \eta) V_{\beta c}^{\alpha} \cdot \Lambda_{\beta}^{\beta} \circ h \circ \pi \cdot M_c^c \circ \pi, \\ (\rho, \eta) V_{bc}^a &= M_a^a \circ \pi \cdot (\rho, \eta) V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned} \tag{6.3}$$

The components of a distinguished linear connection (H, V) verify the change relations:

$$\begin{aligned} H_{jk}^{i'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \pi \cdot \left[\frac{\delta}{\delta x^k} \left(\frac{\partial x^i}{\partial x^j} \circ \pi \right) + H_{jk}^i \cdot \frac{\partial x^j}{\partial x^j} \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ H_{bk}^a &= M_a^a \circ \pi \cdot \left[\frac{\delta}{\delta x^k} (M_b^a \circ \pi) + H_{bk}^a \cdot M_b^b \circ \pi \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \pi, \\ V_{jc}^{i'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \pi \cdot V_{jc}^i \frac{\partial x^j}{\partial x^j} \circ \pi \cdot M_c^c \circ \pi, \\ V_{bc}^a &= M_a^a \circ \pi \cdot V_{bc}^a \cdot M_b^b \circ \pi \cdot M_c^c \circ \pi. \end{aligned} \tag{6.3}'$$

Example 6.1 If (E, π, M) is a vector bundle endowed with the (ρ, η) -connection $(\rho, \eta) \Gamma$, then the local real functions

$$\left(\frac{\partial(\rho, \eta) \Gamma_{\gamma}^a}{\partial y^b}, \frac{\partial(\rho, \eta) \Gamma_{\gamma}^a}{\partial y^b}, 0, 0 \right) \tag{6.4}$$

are the components of a distinguished linear (ρ, η) -connection for $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, which will be called the *Berwald linear (ρ, η) -connection*.

The Berwald linear (Id_{TM}, Id_M) -connection will be called the *Berwald linear connection*.

Theorem 6.2 *If the generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is endowed with a distinguished linear (ρ, η) -connection $((\rho, \eta)H, (\rho, \eta)V)$, then, for any*

$$X = \tilde{Z}^\alpha \tilde{\delta}_\alpha + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr}((\rho, \eta)TE, (\rho, \eta)\tau_E, E),$$

we obtain the formula:

$$\begin{aligned} (\rho, \eta) D_X \left(T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\ \left. \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} \right) = \\ = \tilde{Z}^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \\ \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s} + Y^c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \\ \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta \tilde{y}^{b_1} \otimes \dots \otimes \delta \tilde{y}^{b_s}, \end{aligned}$$

where

$$\begin{aligned} T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} = \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_\gamma \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ + (\rho, \eta) H_{\alpha \gamma}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha \gamma}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} \\ - (\rho, \eta) H_{\beta_1 \gamma}^\beta T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^\beta T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\ + (\rho, \eta) H_{a \gamma}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) H_{a \gamma}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\ - (\rho, \eta) H_{b_1 \gamma}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{b_s \gamma}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \end{aligned}$$

and

$$\begin{aligned} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |_c = \Gamma(\tilde{\rho}, Id_E) \left(\dot{\tilde{\partial}}_c \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\ + (\rho, \eta) V_{\alpha c}^{\alpha_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_{\alpha c}^{\alpha_p} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} a_1 \dots a_r} - \\ - (\rho, \eta) V_{\beta_1 c}^\beta T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q c}^\beta T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\ + (\rho, \eta) V_{ac}^{a_1} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_2 \dots a_r} + \dots + (\rho, \eta) V_{ac}^{a_r} T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} - \\ - (\rho, \eta) V_{b_1 c}^b T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{b_s c}^b T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}. \end{aligned}$$

Definition 6.2 We assume that $(E, \pi, M) = (F, \nu, N)$.

If $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) and

$$\left((\rho, \eta) H_{bc}^a, (\rho, \eta) \tilde{H}_{bc}^a, (\rho, \eta) V_{bc}^a, (\rho, \eta) \tilde{V}_{bc}^a \right)$$

are the components of a distinguished linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ such that

$$(\rho, \eta) H_{bc}^a = (\rho, \eta) \tilde{H}_{bc}^a \text{ and } (\rho, \eta) V_{bc}^a = (\rho, \eta) \tilde{V}_{bc}^a,$$

then we will say that the generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is endowed with a normal distinguished linear (ρ, η) -connection on components $((\rho, \eta)H_{bc}^a, (\rho, \eta)V_{bc}^a)$.

The components of a normal distinguished linear (Id_{TM}, Id_M) -connection (H, V) will be denoted (H_{jk}^i, V_{jk}^i) .

7 Mechanical systems

We consider the following diagram:

$$(7.1) \quad \begin{array}{ccc} E & & (E, [\cdot, \cdot]_{E,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{h} & M \end{array}$$

where $((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta))$ is a generalized Lie algebroid.

Definition 7.1 A triple

$$(7.2) \quad ((E, \pi, M), F_e, (\rho, \eta)\Gamma),$$

where

$$(7.3) \quad F_e = F^a \frac{\partial}{\partial y^a} \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$$

is an external force and $(\rho, \eta)\Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) , will be called *mechanical (ρ, η) -system*.

Definition 7.2 A smooth *Lagrange fundamental function* on the vector bundle (E, π, M) is a mapping $E \xrightarrow{L} \mathbb{R}$ which satisfies the following conditions:

1. $L \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $L \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M) .

Let L be a Lagrangian defined on the total space of the vector bundle (E, π, M) .

If (U, s_U) is a local vector $(m+r)$ -chart for (E, π, M) , then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(7.4) \quad \begin{aligned} L_i &\stackrel{put}{=} \frac{\partial L}{\partial x^i} \stackrel{put}{=} \frac{\partial}{\partial x^i}(L) & L_{ib} &\stackrel{put}{=} \frac{\partial^2 L}{\partial x^i \partial y^b} \stackrel{put}{=} \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial y^b}(L) \right) \\ L_a &\stackrel{put}{=} \frac{\partial L}{\partial y^a} \stackrel{put}{=} \frac{\partial}{\partial y^a}(L) & L_{ab} &\stackrel{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \stackrel{put}{=} \frac{\partial}{\partial y^a} \left(\frac{\partial}{\partial y^b}(L) \right). \end{aligned}$$

Definition 7.3 If for any vector local $m+r$ -chart (U, s_U) of (E, π, M) , we have:

$$(7.5) \quad \text{rank} \|L_{ab}(u_x)\| = r,$$

for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$, then we will say that the Lagrangian L is regular.

Proposition 7.1 If the Lagrangian L is regular, then for any vector local $m+r$ -chart (U, s_U) of (E, π, M) , we obtain the real functions \tilde{L}^{ab} locally defined by

$$(7.6) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{L}^{ab}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{L}^{ab}(u_x) \end{array},$$

where $\|\tilde{L}^{ab}(u_x)\| = \|L_{ab}(u_x)\|^{-1}$, for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 7.4 A smooth *Finsler fundamental function* on the vector bundle (E, π, M) is a mapping $E \xrightarrow{F} \mathbb{R}_+$ which satisfies the following conditions:

1. $F \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $F \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M) ;
3. F is positively 1-homogenous on the fibres of vector bundle (E, π, M) ;
4. For any vector local $m + r$ -chart (U, s_U) of (E, π, M) , the hessian:

$$(7.7) \quad \|F_{ab}^2(u_x)\|$$

is positively define for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 7.5 If L (respectively F) is a smooth Lagrange (respectively Finsler function), then the triple

$$((E, \pi, M), F_e, L) \quad (\text{respectively } ((E, \pi, M), F_e, F))$$

where $F_e = F^a \frac{\partial}{\partial \tilde{y}^a} \in \Gamma(V(\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is an external force, is called *Lagrange mechanical (ρ, η) -system* and *Finsler mechanical (ρ, η) -system*, respectively.

Any Lagrange mechanical (Id_{TM}, Id_M) -system and any Finsler mechanical (Id_{TM}, Id_M) -system will be called *Lagrange mechanical system* and *Finsler mechanical system*, respectively.

8 (ρ, η) -semisprays and (ρ, η) -sprays for mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, (\rho, \eta)\Gamma)$ be an arbitrary mechanical (ρ, η) -system.

Definition 8.1 The vertical section $\mathbb{C} = y^a \tilde{\partial}_a$ will be called the *Liouville section*.

A section $S \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ will be called (ρ, η) -*semispray* if there exists an almost tangent structure e such that $e(S) = \mathbb{C}$.

Let $g \in \mathbf{Man}(E, E)$ be such that (g, h) is a locally invertible \mathbf{B}^V -morphism of (E, π, M) source and target.

Theorem 8.1 *The section*

$$(8.1) \quad S = (g_b^a \circ h \circ \pi) y^b \tilde{\partial}_a - 2 \left(G^a - \frac{1}{4} F^a \right) \tilde{\partial}_a$$

is a (ρ, η) -semispray such that the real local functions G^a , $a \in \overline{1, n}$, satisfy the following conditions

$$(8.2) \quad \begin{aligned} (\rho, \eta) \Gamma_c^a &= (\tilde{g}_c^b \circ h \circ \pi) \frac{\partial(G^a - \frac{1}{4} F^a)}{\partial y^b} \\ &\quad - \frac{1}{2} (g_e^d \circ h \circ \pi) y^e L_{dc}^f \left(\tilde{g}_f^a \circ h \circ \pi \right) \\ &\quad + \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad - \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_e^a \circ h \circ \pi)}{\partial x^i} \end{aligned}$$

In addition, we remark that the local real functions

$$\begin{aligned}
(8.3) \quad (\rho, \eta) \mathring{\Gamma}_c^a &= (\tilde{g}_c^b \circ h \circ \pi) \frac{\partial G^a}{\partial y^b} \\
&\quad - \frac{1}{2} (g_e^d \circ h \circ \pi) y^e L_{dc}^b (\tilde{g}_b^a \circ h \circ \pi) \\
&\quad + \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\
&\quad - \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial (\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i}
\end{aligned}$$

are the components of a (ρ, η) -connection $(\rho, \eta) \mathring{\Gamma}$ for the vector bundle (E, π, M) .

The (ρ, η) -semispray S will be called the *canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \mathring{\Gamma})$ and from locally invertible $\mathbf{B}^\mathbf{V}$ -morphism (g, h)* .

Proof. We consider the **Mod**-endomorphism

$$\begin{aligned}
\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) &\xrightarrow{\mathbb{P}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\
X &\longmapsto \mathcal{J}_{(g, h)}[S, X]_{(\rho, \eta) TE} - [S, \mathcal{J}_{(g, h)} X]_{(\rho, \eta) TE}.
\end{aligned}$$

Let $X = Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a$ be an arbitrary section. Since

$$\begin{aligned}
[S, X]_{(\rho, \eta) TE} &= \left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta) TE} + \left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} \\
&\quad - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta) TE} - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta) TE}
\end{aligned}$$

and

$$\begin{aligned}
\left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^c}{\partial x^i} \tilde{\partial}_c \\
&\quad - Z^b (\rho_b^j \circ h \circ \pi) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e \tilde{\partial}_c \\
&\quad + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c \tilde{\partial}_c, \\
\left[(g_e^a \circ h \circ \pi \cdot y^e) \tilde{\partial}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Y^c}{\partial x^i} \dot{\tilde{\partial}}_c \\
&\quad - Y^b g_b^c \dot{\tilde{\partial}}_c, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b \tilde{\partial}_b \right]_{(\rho, \eta) TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} \tilde{\partial}_c \\
&\quad - 2 Z^b \rho_b^j \circ h \circ \pi \frac{\partial (G^c - \frac{1}{4} F^c)}{\partial x^j} \dot{\tilde{\partial}}_c, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Y^b \dot{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Y^c}{\partial y^a} \dot{\tilde{\partial}}_c - 2 Y^b \frac{\partial (G^c - \frac{1}{4} F^c)}{\partial y^b} \dot{\tilde{\partial}}_c,
\end{aligned}$$

it results that

$$\begin{aligned}
(P_1) \quad \mathcal{J}_{(g,h)} [S, X]_{(\rho,\eta)TE} &= (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^c}{\partial x^i} (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\rho_b^j \circ h \circ \pi) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d - Y^d \dot{\tilde{\partial}}_d \\
&\quad - 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d.
\end{aligned}$$

Since

$$\begin{aligned}
[S, \mathcal{J}_{(g,h)} X]_{(\rho,\eta)TE} &= \left[(g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho,\eta)TE} \\
&\quad - \left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho,\eta)TE}
\end{aligned}$$

and

$$\begin{aligned}
\left[(g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho,\eta)TE} &= -Z^d \tilde{\partial}_d + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^b}{\partial x^i} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d, \\
\left[2 \left(G^a - \frac{1}{4} F^a \right) \dot{\tilde{\partial}}_a, Z^b (\tilde{g}_b^c \circ h \circ \pi) \dot{\tilde{\partial}}_c \right]_{(\rho,\eta)TE} &= 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^a} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d
\end{aligned}$$

it results that

$$\begin{aligned}
(P_2) \quad [S, \mathcal{J}_{(g,h)} X]_{(\rho,\eta)TE} &= -Z^d \tilde{\partial}_d + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) \frac{\partial Z^b}{\partial x^i} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\
&\quad - 2 \left(G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^a} (\tilde{g}_b^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d.
\end{aligned}$$

Using equalities (P_1) and (P_2) , we obtain:

$$\begin{aligned}
\mathbb{P} \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) &= Z^a \tilde{\partial}_a - Y^d \dot{\tilde{\partial}}_d + (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\rho_b^j \circ h \circ \pi) \frac{\partial (g_e^c \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\
&\quad + (g_e^a \circ h \circ \pi) y^e (\rho_a^i \circ h \circ \pi) Z^b \frac{\partial (\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\
&\quad - Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial 2 (G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d
\end{aligned}$$

After some calculations, it results that \mathbb{P} is an almost product structure. Using the equalities (5.1.2) and (5.2.2) it results that

$$\mathbb{P} \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) = (Id - 2(\rho, \eta) \Gamma) \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right),$$

for any $Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ and we obtain

$$\begin{aligned} (\rho, \eta) \Gamma \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) &= Y^d \dot{\tilde{\partial}}_d - \frac{1}{2} (g_e^a \circ h \circ \pi) y^e Z^b L_{ab}^c (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\ &\quad + \frac{1}{2} Z^b \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial(g_e^c \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_c^d \circ h \circ \pi) \dot{\tilde{\partial}}_d \\ &\quad - \frac{1}{2} (g_e^a \circ h \circ \pi) y^e \left(\rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial(\tilde{g}_b^d \circ h \circ \pi)}{\partial x^i} \dot{\tilde{\partial}}_d \\ &\quad + Z^b (\tilde{g}_b^c \circ h \circ \pi) \frac{\partial(G^d - \frac{1}{4} F^d)}{\partial y^c} \dot{\tilde{\partial}}_d. \end{aligned}$$

Since

$$(\rho, \eta) \Gamma \left(Z^a \tilde{\partial}_a + Y^a \dot{\tilde{\partial}}_a \right) = (Y^d + (\rho, \eta) \Gamma_b^d Z^b) \dot{\tilde{\partial}}_d$$

it results the relations (8.3). In addition, since

$$(\rho, \eta) \mathring{\Gamma}_c^a = (\rho, \eta) \Gamma_c^a + \frac{1}{4} \tilde{g}_c^d \circ h \circ \pi \frac{\partial F^a}{\partial y^d}$$

and

$$\begin{aligned} (\rho, \eta) \mathring{\Gamma}_c^{a'} &= (\rho, \eta) \Gamma_c^{a'} + \frac{1}{2} \tilde{g}_c^b \circ h \circ \pi \frac{\partial F^{a'}}{\partial y^b} \\ &= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + (\rho, \eta) \Gamma_c^a \right) M_c^c \circ h \circ \pi \\ &\quad + M_a^{a'} \circ \pi \left(\frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \right) M_c^c \circ h \circ \pi \\ &= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + \left((\rho, \eta) \Gamma_c^a + \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \right) \right) M_c^c \circ h \circ \pi \\ &= M_a^{a'} \circ \pi \left(\rho_c^i \circ h \circ \pi \cdot \frac{\partial M_b^a}{\partial x^i} y^b + (\rho, \eta) \mathring{\Gamma}_c^a \right) M_c^c \circ h \circ \pi \end{aligned}$$

it results the conclusion of the theorem. q.e.d.

Remark 8.1 If $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to connection Γ which is not the same canonical semispray presented by I. Bucataru and R. Miron in [5].

In particular, if $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e = 0$, then we obtain the classical canonical semispray associated to connection Γ .

Using *Theorem 8.1*, we obtain the following:

Theorem 8.2 *The following properties hold good:*

1° Since $\mathring{\tilde{\delta}}_c = \tilde{\delta}_c - (\rho, \eta) \mathring{\Gamma}_c^a \dot{\tilde{\partial}}_a$, $c \in \overline{1, r}$, it results that

$$(8.4) \quad \mathring{\tilde{\delta}}_c = \tilde{\delta}_c - \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \cdot \frac{\partial F^a}{\partial y^b} \dot{\tilde{\partial}}_a, \quad c \in \overline{1, r}.$$

2° Since $\delta \tilde{y}^a = (\rho, \eta) \dot{\Gamma}_c^a d\tilde{z}^c + d\tilde{y}^a$, it results that

$$(8.5) \quad \delta \tilde{y}^a = \delta y^a + \frac{1}{4} \tilde{g}_c^b \circ h \circ \pi \frac{\partial F^a}{\partial y^b} d\tilde{z}^c, \quad a \in \overline{1, r}.$$

Theorem 8.3 *The real local functions*

$$(8.6) \quad \left(\frac{\partial(\rho, \eta) \Gamma_c^a}{\partial y^b}, \frac{\partial(\rho, \eta) \dot{\Gamma}_c^a}{\partial y^b}, 0, 0 \right), \quad a, b, c \in \overline{1, r},$$

and

$$(8.6)' \quad \left(\frac{\partial(\rho, \eta) \dot{\Gamma}_c^a}{\partial y^b}, \frac{\partial(\rho, \eta) \Gamma_c^a}{\partial y^b}, 0, 0 \right), \quad a, b, c \in \overline{1, r},$$

respectively, are the coefficients to a normal Berwald linear (ρ, η) -connection for the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Theorem 8.4 *The tensor of integrability of the (ρ, η) -connection $(\rho, \eta) \dot{\Gamma}$ is as follows:*

$$(8.7) \quad \begin{aligned} (\rho, \eta, h) \mathring{\mathbb{R}}_{cd}^a &= (\rho, \eta, h) \mathbb{R}_{cd}^a + \frac{1}{4} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \Big|_c - \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \Big|_d \right) \\ &+ \frac{1}{16} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^b}{\partial y^e} \tilde{g}_c^f \circ h \circ \pi \frac{\partial^2 F^a}{\partial y^b \partial y^f} - \tilde{g}_c^f \circ h \circ \pi \frac{\partial F^b}{\partial y^f} \tilde{g}_d^e \circ h \circ \pi \frac{\partial^2 F^a}{\partial y^b \partial y^e} \right) \\ &+ \frac{1}{4} \left(L_{cd}^f \circ h \circ \pi \right) \left(\tilde{g}_f^e \circ h \circ \pi \right) \frac{\partial F^a}{\partial y^e}, \end{aligned}$$

where $|_c$ is the h -covariant derivation with respect to the normal Berwald linear (ρ, η) -connection (8.6).

Proof. Since

$$\begin{aligned} (\rho, \eta, h) \mathring{\mathbb{R}}_{cd}^a &= \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_c \right) \left((\rho, \eta) \dot{\Gamma}_d^a \right) - \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_d \right) \left((\rho, \eta) \dot{\Gamma}_c^a \right) \\ &+ L_{cd}^e \circ h \circ (h \circ \pi) (\rho, \eta) \dot{\Gamma}_e^a, \end{aligned}$$

and

$$\begin{aligned} \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_c \right) \left((\rho, \eta) \dot{\Gamma}_d^a \right) &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_c \right) ((\rho, \eta) \Gamma_d^a) \\ &+ \frac{1}{4} \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_c \right) \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right) \\ &- \frac{1}{4} \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} ((\rho, \eta) \Gamma_d^a) \\ &- \frac{1}{16} \tilde{g}_c^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} \left(\tilde{g}_d^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right), \\ \Gamma(\tilde{\rho}, Id_E) \left(\overset{\circ}{\tilde{\delta}}_d \right) \left((\rho, \eta) \dot{\Gamma}_c^a \right) &= \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_d \right) ((\rho, \eta) \Gamma_c^a) \\ &+ \frac{1}{4} \Gamma(\tilde{\rho}, Id_E) \left(\tilde{\delta}_d \right) \left(\tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right) \\ &- \frac{1}{4} \tilde{g}_d^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} ((\rho, \eta) \Gamma_c^a) \\ &- \frac{1}{16} \tilde{g}_d^e \circ h \circ \pi \frac{\partial F^f}{\partial y^e} \frac{\partial}{\partial y^f} \left(\tilde{g}_c^e \circ h \circ \pi \frac{\partial F^a}{\partial y^e} \right), \end{aligned}$$

$$\begin{aligned} L_{cd}^e \circ h \circ \pi \cdot (\rho, \eta) \overset{\circ}{\Gamma}_e^a &= L_{cd}^e \circ h \circ \pi \cdot (\rho, \eta) \Gamma_e^a \\ &+ L_{cd}^e \circ h \circ \pi \cdot \left(\tilde{g}_e^f \circ h \circ \pi \frac{\partial F^a}{\partial y^f} \right) \end{aligned}$$

it results the conclusion of the theorem. q.e.d.

Proposition 8.1 *If S is the canonical (ρ, η) -semispray associated to the mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from \mathbf{B}^v -morphism (g, h) , then*

$$(8.8) \quad 2G^a = 2G^a M_a^a \circ h \circ \pi - (g_b^a \circ h \circ \pi) y^b (\rho_a^i \circ h \circ \pi) \frac{\partial y^a}{\partial x^i}.$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\| \begin{array}{cc} M_a^a \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial M_a^a \circ \pi}{\partial x^i} y^a & M_a^a \circ \pi \end{array} \right\| = \left\| \begin{array}{cc} M_a^a \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial y^a}{\partial x^i} & M_a^a \circ \pi \end{array} \right\|$$

and

$$\left\| \begin{array}{cc} M_a^a \circ h \circ \pi & 0 \\ \rho_a^i \circ (h \circ \pi) \frac{\partial y^a}{\partial x^i} & M_a^a \circ \pi \end{array} \right\| \cdot \begin{pmatrix} (g_b^a \circ h \circ \pi) y^b \\ -2 \left(G^a - \frac{1}{4} F^a \right) \end{pmatrix} = \begin{pmatrix} (g_b^a \circ h \circ \pi) y^b \\ -2 \left(G^a - \frac{1}{4} F^a \right) \end{pmatrix},$$

the conclusion results immediately. q.e.d.

In the following, we consider a differentiable curve $I \xrightarrow{c} M$ and its (g, h) -lift \dot{c} .

Definition 8.3 If it verifies the following equality:

$$(8.9) \quad \frac{d\dot{c}(t)}{dt} = \Gamma(\tilde{\rho}, Id_E) S(\dot{c}(t)),$$

then we say that the curve \dot{c} is an integral curve of the (ρ, η) -semispray S of the mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$,

Theorem 8.5 *All (g, h) -lifts solutions of the equations:*

$$(8.10) \quad \frac{dy^a(t)}{dt} + 2G^a \circ u(c, \dot{c})(x(t)) = \frac{1}{2} F^a \circ u(c, \dot{c})(x(t)), \quad a \in \overline{1, n},$$

where $x(t) = (\eta \circ h \circ c)(t)$, are integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^v -morphism (g, h) .

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt} = \Gamma(\tilde{\rho}, Id_E) S(\dot{c}(t))$$

is equivalent to

$$\begin{aligned} & \frac{d}{dt}((\eta \circ h \circ c)^i(t), y^a(t)) \\ &= \left(\rho_a^i \circ \eta \circ h \circ c(t) g_b^a \circ h \circ c(t) y^b(t), -2 \left(G^a - \frac{1}{4} F^a \right) ((\eta \circ h \circ c)^i(t), y^a(t)) \right), \end{aligned}$$

it results

$$\begin{aligned} & \frac{dy^a(t)}{dt} + 2G^a(x^i(t), y^a(t)) = \frac{1}{2} F^a(x^i(t), y^a(t)), \quad a \in \overline{1, n}, \\ & \frac{dx^i(t)}{dt} = \rho_a^i \circ \eta \circ h \circ c(t) g_b^a \circ h \circ c(t) y^b(t), \end{aligned}$$

where $x^i(t) = (\eta \circ h \circ c)^i(t)$.

q.e.d.

Definition 8.4 If S is a (ρ, η) -semispray, then the vector field

$$(8.11) \quad [\mathbb{C}, S]_{(\rho, \eta)TE} - S$$

will be called the *derivation of (ρ, η) -semispray S* .

The (ρ, η) -semispray S will be called (ρ, η) -*spray* if the following conditions are verified:

1. $S \circ 0 \in C^1$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ, η) -semispray S will be called *quadratic (ρ, η) -spray* if there are verified the following conditions:

1. $S \circ 0 \in C^2$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho, \eta) = (id_{TM}, Id_M)$ and $(g, h) = (Id_E, Id_M)$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 8.6 If S is the canonical (ρ, η) -spray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) , then

$$(8.12) \quad \begin{aligned} 2(G^a - \frac{1}{4}F^a) &= (\rho, \eta) \Gamma_c^a (g_f^c \circ h \circ \pi) y^f \\ &+ \frac{1}{2} (g_e^d \circ h \circ \pi) y^e (L_{dc}^b \circ h \circ \pi) (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \\ &- \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \\ &+ \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} (g_f^c \circ h \circ \pi) y^f \end{aligned}$$

We obtain the spray

$$(8.13) \quad \begin{aligned} S &= (g_b^a \circ h \circ \pi) y^b \dot{\tilde{\partial}}_a - (\rho, \eta) \Gamma_c^a (g_f^c \circ h \circ \pi) y^f \dot{\tilde{\partial}}_a \\ &- \frac{1}{2} (g_e^d \circ h \circ \pi) y^e (L_{dc}^b \circ h \circ \pi) (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \dot{\tilde{\partial}}_a \\ &+ \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) (g_f^c \circ h \circ \pi) y^f \dot{\tilde{\partial}}_a \\ &- \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} (g_f^c \circ h \circ \pi) y^f \dot{\tilde{\partial}}_a \end{aligned}$$

This spray will be called the *canonical (ρ, η) -spray associated to mechanical system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h)* .

In particular, if $(\rho, \eta) = (id_{TM}, Id_M)$ and $(g, h) = (Id_E, Id_M)$, then we get the *canonical spray associated to connection Γ which is similar with the classical canonical spray associated to connection Γ* .

Proof. Since

$$[\mathbb{C}, S]_{(\rho, \eta)TE} = \left[y^a \dot{\tilde{\partial}}_a, (g_e^b \circ h \circ \pi \cdot y^e) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} - 2 \left[y^a \dot{\tilde{\partial}}_a, (G^b - \frac{1}{4}F^b) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE},$$

$$\left[y^a \dot{\tilde{\partial}}_a, (g_e^b \circ h \circ \pi \cdot y^e) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} = \left(g_e^b \circ h \circ \pi \cdot y^e \right) \dot{\tilde{\partial}}_b$$

and

$$\left[y^a \dot{\tilde{\partial}}_a, \left(G^b - \frac{1}{4} F^b \right) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} = y^a \frac{\partial \left(G^b - \frac{1}{4} F^b \right)}{\partial y^a} \dot{\tilde{\partial}}_b - \left(G^b - \frac{1}{4} F^b \right) \dot{\tilde{\partial}}_b$$

it results that

$$(S_1) \quad [\mathbb{C}, S]_{(\rho, \eta)TE} - S = 2 \left(-y^f \frac{\partial \left(G^a - \frac{1}{4} F^a \right)}{y^f} + 2 \left(G^a - \frac{1}{4} F^a \right) \right) \dot{\tilde{\partial}}_a$$

Using equality (8.3), it results that

$$(S_2) \quad \begin{aligned} \frac{\partial \left(G^a - \frac{1}{4} F^a \right)}{y^f} &= (\rho, \eta) \Gamma_c^a \left(g_f^c \circ h \circ \pi \right) \\ &+ \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) \\ &- \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) \\ &+ \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e \left(\rho_b^i \circ h \circ \pi \right) \frac{\partial (\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} \left(g_f^c \circ h \circ \pi \right) \end{aligned}$$

Using equalities (S₁) and (S₂), it results the conclusion of the theorem. *q.e.d.*

Theorem 8.7 *All (g, h)-lifts solutions of the following system of equations:*

$$(8.14) \quad \begin{aligned} &\frac{dy^a}{dt} + (\rho, \eta) \Gamma_c^a \left(g_f^c \circ h \circ \pi \right) y^f \\ &+ \frac{1}{2} \left(g_e^d \circ h \circ \pi \right) y^e \left(L_{dc}^b \circ h \circ \pi \right) \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \\ &- \frac{1}{2} \left(\rho_c^j \circ h \circ \pi \right) \frac{\partial (g_e^b \circ h \circ \pi)}{\partial x^j} y^e \left(\tilde{g}_b^a \circ h \circ \pi \right) \left(g_f^c \circ h \circ \pi \right) y^f \\ &+ \frac{1}{2} \left(g_e^b \circ h \circ \pi \right) y^e \left(\rho_b^i \circ h \circ \pi \right) \frac{\partial (\tilde{g}_c^a \circ h \circ \pi)}{\partial x^i} \left(g_f^c \circ h \circ \pi \right) y^f = 0, \end{aligned}$$

are the integral curves of canonical (ρ, η) -spray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .

9 A Lagrangian formalism for Lagrange mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, L)$ be an arbitrarily Lagrange mechanical (ρ, η) -system.

Let $(d\tilde{z}^a, d\tilde{y}^a)$ be the natural dual (ρ, η) -base of the natural (ρ, η) -base $\left(\dot{\tilde{\partial}}_\alpha, \dot{\tilde{\partial}}_a \right)$.

It is very important to remark that the 1-forms $d\tilde{z}^a, d\tilde{y}^a$, $a \in \overline{1, p}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

$$(d\tilde{z}^a) \neq d^{(\rho, \eta)TE}(\tilde{z}^a),$$

where $d^{(\rho,\eta)TE}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}(E)$ -algebra

$$(\Lambda((\rho, \eta)TE, (\rho, \eta)\tau_E, E), +, \cdot, \wedge).$$

Let L be a regular Lagrangian and let (g, h) be a locally invertible \mathbf{B}^v -morphism of (E, π, M) source and target.

Definition 9.1 The 1-form

$$(9.1) \quad \theta_L = (\tilde{g}_a^e \circ h \circ \pi \cdot L_e) d\tilde{z}^a$$

will be called the 1-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible \mathbf{B}^v -morphism (g, h) .

We obtain easily:

$$(9.2) \quad \theta_L(\tilde{\partial}_a) = \tilde{g}_b^e \circ h \circ \pi \cdot L_e, \quad \theta_L(\dot{\tilde{\partial}}_b) = 0.$$

Definition 9.2 The 2-form

$$\omega_L = d^{(\rho,\eta)TE}\theta_L$$

will be called the 2-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible \mathbf{B}^v -morphism (g, h) .

By the definition of $d^{(\rho,\eta)TE}$, we obtain:

$$(9.3) \quad \begin{aligned} \omega_L(U, V) &= \Gamma(\tilde{\rho}, Id_E)(U)(\theta_L(V)) \\ &\quad - \Gamma(\tilde{\rho}, Id_E)(V)(\theta_L(U)) - \theta_L([U, V]_{(\rho,\eta)TE}), \end{aligned}$$

for any $U, V \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Definition 9.3 The real function

$$(9.4) \quad \mathcal{E}_L = y^a L_a - L$$

will be called the energy of regular Lagrangian L .

Theorem 9.1 The equation

$$(9.5) \quad i_S(\omega_L) = -d^{(\rho,\eta)TE}(\mathcal{E}_L), \quad S \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E),$$

has an unique solution $S_L(g, h)$ of the type:

$$(9.6) \quad (g_e^a \circ h \circ \pi) y^e \tilde{\partial}_a - 2(G^a - \frac{1}{4}F^a) \dot{\tilde{\partial}}_a,$$

where

$$(9.7) \quad -2(G^a - \frac{1}{4}F^a) = E_b(L, g, h) \tilde{L}^{be} (g_e^a \circ h \circ \pi)$$

and

$$(9.8) \quad \begin{aligned} E_b(L, g, h) &= (\rho_b^i \circ h \circ \pi) L_i - (\rho_b^i \circ h \circ \pi) y^a L_{ia} \\ &\quad - \left(g_f^d \circ h \circ \pi \right) y^f (\rho_d^i \circ h \circ \pi) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial x^i} \\ &\quad + \left(g_f^d \circ h \circ \pi \right) y^f (\rho_b^i \circ h \circ \pi) \frac{\partial((\tilde{g}_d^e \circ h \circ \pi) L_e)}{\partial x^i} \\ &\quad + \left(g_f^d \circ h \circ \pi \right) y^f (L_{db}^c \circ h \circ \pi) (\tilde{g}_c^e \circ h \circ \pi) L_e \end{aligned}$$

$S_L(g, h)$ will be called the canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) .

Proof. We obtain that

$$i_S(\omega_L) = -d^{(\rho, \eta)TE}(\mathcal{E}_L)$$

if and only if

$$\omega_L(S, X) = -\Gamma(\tilde{\rho}, Id_E)(X)(\mathcal{E}_L),$$

for any $X \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

Particularly, we obtain:

$$\omega_L(S, \tilde{\partial}_b) = -\Gamma(\tilde{\rho}, Id_E)(\tilde{\partial}_b)(\mathcal{E}_L).$$

If we expand this equality, we obtain

$$\begin{aligned} & (g_f^d \circ h \circ \pi) y^f \left[(\rho_b^i \circ h \circ \pi) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi)L_e)}{\partial x^i} - (\rho_b^i \circ h \circ \pi) \frac{\partial((\tilde{g}_d^e \circ h \circ \pi)L_e)}{\partial x^i} \right. \\ & \left. - (L_{ab}^c \circ h \circ \pi)(\tilde{g}_c^e \circ h \circ \pi)L_e \right] - 2 \left(G^a - \frac{1}{4} F^a \right) (\tilde{g}_a^e \circ h \circ \pi) \cdot L_{eb} \\ & = \rho_b^i \circ h \circ \pi \cdot L_i - (\rho_b^i \circ h \circ \pi) \frac{\partial(y^a L_a)}{\partial x^i}. \end{aligned}$$

After some calculations, we obtain the conclusion of the theorem. *q.e.d.*

Remark 9.1 If $F_e = 0$ and $\eta = Id_M$, then

$$E_b(L, Id_E, Id_M) = (\rho_b^i \circ \pi) L_i - (\rho_b^i \circ \pi) y^d L_{id} + y^d (L_{db}^c \circ \pi) L_c$$

and $S_L(Id_E, Id_M) \stackrel{put}{=} S_L$ is the canonical ρ -semispray associated to regular Lagrangian L which is similar with the semispray presented in [9] by M. de Leon, J. Marrero and E. Martinez. (see also [11])

In addition, if $F_e \neq 0$ and $(\rho, \eta) = (Id_{TM}, Id_M)$, then $S_L(Id_E, Id_M) \stackrel{put}{=} S_L$ will be called the canonical semispray which is similar with the semispray presented by I. Bucataru and R. Miron in [5].

In particular, if $F_e = 0$ and $(\rho, \eta) = (Id_{TM}, Id_M)$, then $S_L(Id_M, Id_E) \stackrel{put}{=} S_L$ will be called the canonical semispray which is similar with the canonical semispray presented by R. Miron and M. Anastasiei in [13]. (see also [14, 15])

Theorem 9.2 *If $S_L(g, h)$ is the canonical (ρ, η) -semispray associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) , then the real local functions*

$$\begin{aligned} (\rho, \eta) \Gamma_c^a &= -\frac{1}{2} (\tilde{g}_c^d \circ h \circ \pi) \frac{\partial(E_b(L, g, h) \tilde{L}^{be}(g_e^a \circ h \circ \pi))}{\partial y^d} \\ &\quad - \frac{1}{2} (g_e^d \circ h \circ \pi) y^e \left(L_{dc}^f \circ h \circ \pi \right) (\tilde{g}_f^a \circ h \circ \pi) \\ &\quad + \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g_e^b \circ h \circ \pi)}{\partial x^j} y^e (\tilde{g}_b^a \circ h \circ \pi) \\ &\quad - \frac{1}{2} (g_e^b \circ h \circ \pi) y^e (\rho_b^i \circ h \circ \pi) \frac{\partial(\tilde{g}_a^e \circ h \circ \pi)}{\partial x^i} \end{aligned} \tag{9.9}$$

are the components of a (ρ, η) -connection $(\rho, \eta) \Gamma$ for the vector bundle (E, π, M) which will be called the (ρ, η) -connection associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) .

In particular, if $\eta = h = Id_M$ and $g = Id_E$, then we obtain

$$(9.9') \quad \rho \Gamma_c^a = -\frac{1}{2} \frac{\partial \left(E_b(L, Id_E, Id_M) \tilde{L}^{ba} \right)}{\partial y^c} - \frac{1}{2} y^b L_{bc}^a \circ \pi.$$

Theorem 9.3 *The parallel (g, h) -lifts with respect to (ρ, η) -connection $(\rho, \eta) \Gamma$ are the integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) .*

Definition 9.4 The equations

$$(9.10) \quad \frac{dy^a(t)}{dt} - \left(E_b(L, g, h) \tilde{L}^{be} (g_e^a \circ h \circ \pi) \right) \circ u(c, \dot{c})(x(t)) = 0,$$

where $x(t) = \eta \circ h \circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h)* .

The equations

$$(9.10') \quad \frac{dy^a(t)}{dt} - \left(E_b(L, Id_E, Id_M) \tilde{L}^{ba} \right) \circ u(c, \dot{c})(x(t)) = 0,$$

where $x(t) = c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$* .

Remark 9.1 The integral curves of the canonical (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, L)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) are the (g, h) -lifts solutions for the equations of Euler-Lagrange type (9.10).

Using our theory, we obtain the following

Theorem 9.4 *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g, h) -lifts are solutions for the equations of Euler-Lagrange type (9.10).*

Therefore, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid $((TM, \tau_M, M), [\cdot]_{TM}, (Id_{TM}, Id_M))$, to an arbitrary (generalized) Lie algebroid $((E, \pi, M), [\cdot]_{E, h}, (\rho, \eta))$.

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